Nonlinear Functional Analysis and Applications
Vol. 12, No. 1 (2007), pp. 119-152

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

THE ADDITIVE APPROXIMATION ON A SEVEN VARIABLE JENSEN TYPE FUNCTIONAL EQUATION

Shuxiang Shan[1] and Jian Wang[2]

[1]Department of Mathematics, Fujian Normal University,
Fuzhou 350007, P. R. China
e-mail: ssxmath@126.com

[2]Department of Mathematics, Fujian Normal University,
Fuzhou 350007, P. R. China
e-mail: wangjian@fjnu.edu.cn

Abstract. We investigate the Hyers-Ulam stability of a 7-variate Jensen type functional equation, getting the corresponding error formulas in which the β-homogeneity of the F-norm is closely associated with the approximate remainder ϕ. Finally, we make sure p_i's area in which the Hyers-Ulam stability is affirmative or negative where p_i for $i = 1, 2, \cdots, 7$ can be different.

1. Introduction

Throughout this paper, let G be a linear space and let E be a real or complex Hausdorff topological vector space. Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and of real numbers, respectively. We assume f be a mapping from G into E. We take the following equations:

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \theta, \tag{1}$$

[0]Received April 28, 2006. Revised October 24, 2006.
[0]2000 Mathematics Subject Classification: 47A62, 39B72.
[0]Keywords: Hyers-Ulam stability, Jensen type functional equation, approximate remainder
[0]Supported by the National Science Foundation of Fujian Province (Z0511023) and the Foundation of Fujian Educational Committee (JA04172).

$$\begin{aligned}
&7f\left(\tfrac{1}{7}\sum_{i=1}^{7}x_i\right)-6\sum_{i=1}^{7}f\left(\tfrac{1}{6}\sum_{\substack{j=1\\ j\neq i}}^{7}x_j\right)+5\sum_{i=1}^{7}f\left(\tfrac{1}{5}\sum_{\substack{j=1\\ j\neq i,i+1}}^{7}x_j\right)\\
&-\sum_{i=1}^{7}f\left(\tfrac{1}{4}\sum_{\substack{j=1\\ j\neq i,i+1,i+2}}^{7}x_j\right)+3\sum_{i=1}^{7}f\left(\tfrac{1}{3}\sum_{j=i}^{i+2}x_j\right)-2\sum_{i=1}^{7}f\left(\tfrac{1}{2}\sum_{j=i}^{i+1}x_j\right)\\
&=\theta \qquad\qquad (x_8=x_1, x_9=x_2)
\end{aligned}\tag{2}$$

as a Jensen equation and a 7-variable Jensen type functional equation, respectively. We define the approximate remainder ϕ by

$$\begin{aligned}
&7f\left(\tfrac{1}{7}\sum_{i=1}^{7}x_i\right)-6\sum_{i=1}^{7}f\left(\tfrac{1}{6}\sum_{\substack{j=1\\ j\neq i}}^{7}x_j\right)+5\sum_{i=1}^{7}f\left(\tfrac{1}{5}\sum_{\substack{j=1\\ j\neq i,i+1}}^{7}x_j\right)\\
&-\sum_{i=1}^{7}f\left(\tfrac{1}{4}\sum_{\substack{j=1\\ j\neq i,i+1,i+2}}^{7}x_j\right)+3\sum_{i=1}^{7}f\left(\tfrac{1}{3}\sum_{j=i}^{i+2}x_j\right)-2\sum_{i=1}^{7}f\left(\tfrac{1}{2}\sum_{j=i}^{i+1}x_j\right)\\
&=\phi(x_1,x_2,\cdots,x_7) \qquad\qquad (x_8=x_1, x_9=x_2)
\end{aligned}\tag{3}$$

for all $x_i\in G, i=1,2,\cdots,7$.

In 1940, Ulam [14] raised the following problem:

Let G be a group and let E be a metric group with the metry $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if a function $h\colon G\to E$ satisfies the inequality

$$d(h(xy),h(x)h(y))<\delta$$

for all $x,y\in G$. Then there exists a homomorphism $H\colon G\to E$ with $d(h(x),H(x))<\varepsilon$ for all $x\in G$?

For Banach spaces Hyers [2] firstly solved this question in 1941. In 1978, Rassias [9] generalized the result of Hyers significantly. The result was further generalized by Rassias [10], Rassias and Šemrl [11], Găvrută [1]. Since then, the stability problems of Jensen equations have been extensively investigated by a number of mathematicians ([18], [5], [6] and [7]).

Trif [13] studied the stability of the Hyers-Ulam-Rassias of the 3-variable Jensen type functional equation

$$\begin{aligned}
&3f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z)\\
=\;&2f\left(\frac{x+y}{2}\right)+2f\left(\frac{y+z}{2}\right)+2f\left(\frac{z+x}{2}\right)
\end{aligned}$$

under the assumption that G and E are a real normed linear space and a real Banach space, respectively.

Wang [18] investigated the stability of the Hyers-Ulam of the 4-variable Jensen type functional equation

$$\begin{aligned}
&4f\left(\frac{x+y+z+w}{4}\right)+2f\left(\frac{z+w}{2}\right)+2f\left(\frac{x+w}{2}\right)+2f\left(\frac{x+y}{2}\right)\\
&+2f\left(\frac{y+z}{2}\right)\\
=&\,3f\left(\frac{y+z+w}{3}\right)+3f\left(\frac{x+z+w}{3}\right)+3f\left(\frac{x+y+w}{3}\right)\\
&+3f\left(\frac{x+y+z}{3}\right)
\end{aligned}$$

under the assumption that G and E are a real linear space and a certain kind of F^*-space, respectively.

In this paper, using approximate remainder we study the Hyers-Ulam stability of Eq.(2), where G, E are a real linear space and a certain kind of F^*-space, respectively. At first we solve Eq.(2), then we work out some theorems associated with the Hyers-Ulam stability of Eq.(2). At last, we give p_i's area in which the Hyers-Ulam stability is affirmative or negative where p_i $(i = 1, 2, \cdots, 7)$ can be different.

2. Solutions of Eq.(2)

From now on, we denote by G a real linear space, by E a real Hausdorff topological vector space if there is no special case. In this section we claim that Eq.(2) is equivalent to Eq.(1). It is well known that if G and E are real linear spaces, a function $f\colon G \to E$ satisfying $f(\theta) = \theta$ is a solution of Eq.(1) if and only if it is additive.

Theorem 1. *A function $f\colon G \to E$ satisfies Eq.(2) for all $x_i \in G$($i = 1, 2, \cdots, 7$) if and only if there exist a constant element $C \in E$ and a unique additive mapping $T\colon G \to E$ such that*

$$f(x) = T(x) + C \quad (\forall x \in G).$$

Proof. Sufficiency: The sufficiency is so obvious that we shall omit it.

Necessity: Set $C = f(\theta)$ and $T(x) = f(x) - C$ for each $x \in G$. Then equalities $T(\theta) = \theta$ and

$$\begin{aligned}
&7T\left(\frac{1}{7}\sum_{i=1}^{7}x_i\right)-6\sum_{i=1}^{7}T\left(\frac{1}{6}\sum_{\substack{j=1\\ j\neq i}}^{7}x_j\right)+5\sum_{i=1}^{7}T\left(\frac{1}{5}\sum_{\substack{j=1\\ j\neq i,i+1}}^{7}x_j\right)\\
&-\sum_{i=1}^{7}T\left(\frac{1}{4}\sum_{\substack{j=1\\ j\neq i,i+1,i+2}}^{7}x_j\right)+3\sum_{i=1}^{7}T\left(\frac{1}{3}\sum_{j=i}^{i+2}x_j\right)-2\sum_{i=1}^{7}T\left(\frac{1}{2}\sum_{j=i}^{i+1}x_j\right)\\
&=\theta \qquad\qquad (x_8=x_1, x_9=x_2)
\end{aligned}\tag{4}$$

hold for all $x_i \in G$ ($i = 1, 2, \cdots, 7$).

Next we shall prove the additivity of T. Let $x \in G$. Substitutions of $x_1 = x_3 = x, x_2 = x_4 = -x, x_5 = x_6 = x_7 = \theta$ in (4) yield

$$\begin{aligned}
&-12\left[T\left(-\tfrac{x}{6}\right)+T\left(\tfrac{x}{6}\right)\right]+5\left[T\left(-\tfrac{x}{5}\right)+T\left(\tfrac{x}{5}\right)\right]-2\left[T\left(-\tfrac{x}{4}\right)+T\left(\tfrac{x}{4}\right)\right]\\
&+6\left[T\left(-\tfrac{x}{3}\right)+T\left(\tfrac{x}{3}\right)\right]-2\left[T\left(-\tfrac{x}{2}\right)+T\left(\tfrac{x}{2}\right)\right]\\
&=\theta.
\end{aligned}\tag{5}$$

Taking $x_1 = x_3 = x_5 = x, x_2 = x_4 = x_6 = -x, x_7 = \theta$ in (4) gets

$$\begin{aligned}
&-18\left[T\left(-\tfrac{x}{6}\right)+T\left(\tfrac{x}{6}\right)\right]+5\left[T\left(-\tfrac{x}{5}\right)+T\left(\tfrac{x}{5}\right)\right]-2\left[T\left(-\tfrac{x}{4}\right)+T\left(\tfrac{x}{4}\right)\right]\\
&+6\left[T\left(-\tfrac{x}{3}\right)+T\left(\tfrac{x}{3}\right)\right]-2\left[T\left(-\tfrac{x}{2}\right)+T\left(\tfrac{x}{2}\right)\right]\\
&=\theta.
\end{aligned}\tag{6}$$

Subtracting (6) from (5) gives

$$6\left[T\left(-\frac{x}{6}\right)+T\left(\frac{x}{6}\right)\right]=\theta, \text{ i.e., } T\left(-\frac{x}{6}\right)=-T\left(\frac{x}{6}\right).$$

Replacing $6x$ by x in the last equality, we obtain

$$T(-x) = -T(x). \tag{7}$$

substituting $x_1 = x_2 = x, x_3 = -2x, x_4 = x_5 = x_6 = x_7 = \theta$ in (4), we get

$$\begin{aligned}
&-6\left[T\left(\tfrac{x}{3}\right)+2T\left(-\tfrac{x}{6}\right)\right]+5\left[T\left(-\tfrac{2x}{5}\right)+T\left(\tfrac{2x}{5}\right)\right]+5\left[T\left(-\tfrac{x}{5}\right)+T\left(\tfrac{x}{5}\right)\right]\\
&-\left[T\left(-\tfrac{x}{4}\right)+T\left(\tfrac{x}{4}\right)\right]-3\left[T\left(-\tfrac{x}{2}\right)+T\left(\tfrac{x}{2}\right)\right]+3\left[T\left(-\tfrac{x}{3}\right)+T\left(\tfrac{x}{3}\right)\right]\\
&+3\left[T\left(-\tfrac{2x}{3}\right)+T\left(\tfrac{2x}{3}\right)\right]-2\left[T(x)+T(-x)\right]\\
&=\theta.
\end{aligned}\tag{8}$$

Combining (7) and (8), we get

$$T\left(\frac{x}{6}\right)=\frac{1}{2}T\left(\frac{x}{3}\right).$$

With $3x$ in place of x in the above equality, we get

$$T\left(\frac{x}{2}\right)=\frac{1}{2}T\left(x\right). \tag{9}$$

So we have $T\left(\frac{x}{4}\right)=\frac{1}{4}T\left(x\right)$. Substituting $x_1=x_2=x_3=x_4=x, x_5=-4x, x_6=x_7=\theta$ in (4), we have

$$\begin{aligned}&-6\left[T\left(\tfrac{4x}{6}\right)+4T\left(-\tfrac{x}{6}\right)\right]+5\left[3T\left(-\tfrac{2x}{5}\right)+T\left(\tfrac{3x}{5}\right)+T\left(\tfrac{4x}{5}\right)+T\left(-\tfrac{x}{5}\right)\right]\\&-\left[T\left(-\tfrac{3x}{4}\right)+T\left(\tfrac{3x}{4}\right)\right]-\left[T\left(-\tfrac{x}{2}\right)+T\left(\tfrac{x}{2}\right)\right]-\left[T\left(-\tfrac{3x}{4}\right)+T(x)+T\left(-\tfrac{x}{4}\right)\right]\\&+3\left[T(x)+T(-x)\right]+3\left[T\left(-\tfrac{2x}{3}\right)+T\left(\tfrac{2x}{3}\right)\right]+3\left[T(x)+T\left(-\tfrac{4x}{3}\right)+T\left(\tfrac{x}{3}\right)\right]\\&-2\left[3T(x)+T\left(-\tfrac{3x}{2}\right)+T\left(-2x\right)+T\left(\tfrac{x}{2}\right)\right]=\theta.\end{aligned} \tag{10}$$

Applying (7) and (9) to (10), we get

$$3\left[T(x)-3T\left(\frac{x}{3}\right)\right]+\frac{5}{4}\left[T(3x)-3T(x)\right]+5\left[T\left(\frac{3x}{5}\right)-3T\left(\frac{x}{5}\right)\right]=\theta. \tag{11}$$

Setting $x_1=x_2=x_3=x$, $x_4=-3x$, $x_5=x_6=x_7=\theta$ in (4), we get

$$\begin{aligned}&-6\left[T\left(-\tfrac{x}{6}\right)+T\left(-\tfrac{x}{2}\right)\right]\\&+5\left[T\left(-\tfrac{2x}{5}\right)+T\left(\tfrac{2x}{5}\right)\right]+5\left[T\left(-\tfrac{2x}{5}\right)+T\left(\tfrac{3x}{5}\right)+T\left(-\tfrac{x}{5}\right)\right]\\&-\left[T\left(-\tfrac{3x}{4}\right)+T\left(\tfrac{3x}{4}\right)\right]-\left[T\left(-\tfrac{x}{4}\right)+T\left(\tfrac{x}{4}\right)\right]-\left[T\left(-\tfrac{x}{2}\right)+T\left(\tfrac{x}{2}\right)\right]\\&+3\left[T(x)+T(-x)\right]+3\left[T\left(-\tfrac{x}{3}\right)+T\left(\tfrac{x}{3}\right)\right]+3\left[T\left(-\tfrac{2x}{3}\right)+T\left(\tfrac{2x}{3}\right)\right]\\&-2\left[T(x)+T(-x)\right]-2\left[T(x)+T\left(-\tfrac{3x}{2}\right)+T\left(\tfrac{x}{2}\right)\right]=\theta.\end{aligned} \tag{12}$$

Applying (7) and (9) to (12), we conclude

$$-3\left[T(x)-3T\left(\frac{x}{3}\right)\right]+\left[T(3x)-3T(x)\right]+5\left[T\left(\frac{3x}{5}\right)-3T\left(\frac{x}{5}\right)\right]=\theta. \tag{13}$$

It follows from (11) and (13) that

$$T(3x)-3T(x)=-24\left[T(x)-3T\left(\frac{x}{3}\right)\right]. \tag{14}$$

By substituting (14) into (13), we have

$$T\left(\frac{3x}{5}\right)-3T\left(\frac{x}{5}\right)=\frac{27}{5}\left[T(x)-3T\left(\frac{x}{3}\right)\right]. \tag{15}$$

Substitutions of $x_1 = x_2 = x_3 = x$, $x_5 = -3x$, $x_4 = x_6 = x_7 = \theta$ in (4) give

$$\begin{aligned}&-6\left[3T\left(-\tfrac{x}{6}\right)+T\left(\tfrac{x}{2}\right)\right]+10\left[T\left(-\tfrac{2x}{5}\right)+T\left(\tfrac{3x}{5}\right)+T\left(-\tfrac{x}{5}\right)\right]\\&-\left[T\left(-\tfrac{x}{2}\right)+T\left(\tfrac{x}{2}\right)\right]-\left[T\left(-\tfrac{3x}{4}\right)+2T\left(\tfrac{3x}{4}\right)+T\left(-\tfrac{x}{4}\right)+T\left(-\tfrac{x}{2}\right)\right]\\&+3\left[T\left(-\tfrac{2x}{3}\right)+T\left(\tfrac{2x}{3}\right)\right]+3\left[2T(-x)+T(x)+T\left(\tfrac{2x}{3}\right)+T\left(\tfrac{x}{3}\right)\right]\\&-4\left[T(x)+T\left(-\tfrac{3x}{2}\right)+T\left(\tfrac{x}{2}\right)\right]=\theta.\end{aligned}\tag{16}$$

Applying (7) and (9) to (16), we get

$$-6\left[T(x)-3T\left(\frac{x}{3}\right)\right]+10\left[T\left(\frac{3x}{5}\right)-3T\left(\frac{x}{5}\right)\right]+\frac{7}{4}\left[T(3x)-3T\left(3x\right)\right]=\theta.\tag{17}$$

Applying (14) and (15) to (17), we get

$$T\left(x\right)-3T\left(\frac{x}{3}\right)=\theta,\quad \text{i.e.,}\quad T\left(\frac{x}{3}\right)=\frac{1}{3}T(x)\tag{18}$$

and so $T\left(\frac{x}{6}\right) = \frac{1}{2}T\left(\frac{x}{3}\right) = \frac{1}{6}T(x)$. By setting $x_1 = x_2 = x_3 = x_4 = x_5 = x$, $x_6 = -5x$, $x_7 = \theta$ in (4), we have

$$\begin{aligned}&-6\left[T\left(\tfrac{5x}{6}\right)+5T\left(-\tfrac{x}{5}\right)\right]+5\left[4T\left(-\tfrac{2x}{5}\right)+T\left(\tfrac{4x}{5}\right)+T(x)+T\left(-\tfrac{x}{5}\right)\right]\\&-\left[3T\left(-\tfrac{3x}{4}\right)+T\left(\tfrac{3x}{4}\right)+2T(x)+T\left(-\tfrac{x}{2}\right)\right]\\&+3\left[3T(x)+T(-x)+T\left(-\tfrac{4x}{3}\right)+T\left(\tfrac{2x}{3}\right)\right]\\&-2\left[4T(x)+T(-2x)+T\left(-\tfrac{5x}{2}\right)+T\left(\tfrac{x}{2}\right)\right]=\theta.\end{aligned}\tag{19}$$

Applying (7), (9) and (18) to (19), we get

$$5\left[T\left(x\right)-5T\left(\frac{x}{5}\right)\right]=\theta,\quad \text{i.e.,}\quad T\left(\frac{x}{5}\right)=\frac{1}{5}T(x).\tag{20}$$

Put $x_1 = x$, $x_2 = x_3 = \cdots = x_7 = \theta$ in (4), then

$$7T(\frac{x}{7})-36T(\frac{x}{6})+25T(\frac{x}{5})-4T(\frac{x}{4})+9T(\frac{x}{3})-4T(\frac{x}{2})=\theta.\tag{21}$$

Applying (7), (9), (18) and (20) to (21), we have

$$7T(\frac{x}{7})-T(x)=\theta,\quad \text{i.e.,}\quad T(\frac{x}{7})=\frac{1}{7}T(x).\tag{22}$$

Substitutions of $T(\frac{x}{7}) = \frac{1}{7}T(x)$, $T(\frac{x}{6}) = \frac{1}{6}T(x)$, $T(\frac{x}{5}) = \frac{1}{5}T(x)$, $T(\frac{x}{3}) = \frac{1}{3}T(x)$, $T(\frac{x}{2}) = \frac{1}{2}T(x)$ in (7) imply

$$T\left(\sum_{i=1}^{7} x_i\right) - \sum_{i=1}^{7} T\left(\sum_{\substack{j=1\\ j\neq i}}^{7} x_j\right) + \sum_{i=1}^{7} T\left(\sum_{\substack{j=1\\ j\neq i,i+1}}^{7} x_j\right)$$

$$-\sum_{i=1}^{7} T\left(\frac{1}{4}\sum_{\substack{j=1\\ j\neq i,i+1,i+2}}^{7} x_j\right) + \sum_{i=1}^{7} T\left(\sum_{j=i}^{i+2} x_j\right) - \sum_{i=1}^{7} T\left(\sum_{j=i}^{i+1} x_j\right) = \theta.$$

If we take $x_1 = x,\ x_2 = y,\ x_3 = -x - y,\ x_4 = x_5 = x_6 = x_7 = \theta$ in the above equality, we get $T\left(x + y\right) = T\left(x\right) + T\left(y\right)$ for arbitrary $x,\ y \in G$. Obviously, T is additive. $\square$

3. Hyers–Ulam–Rassias stability of Eq.(2)

We shall study the Hyers-Ulam stability of Eq.(2).

Theorem 2. *If the approximate remainder ϕ satisfies*

$$\lim_{n\to\infty} \frac{\phi(3^n x_1, 3^n x_2, \cdots, 3^n x_7)}{3^n} = \theta \quad (\ \forall x_i \in G,\ i = 1, 2, \cdots, 7\ ) \tag{23}$$

$$\sum_{k=1}^{\infty} \frac{\Psi(3^k x)}{3^k} = \eta(x) \in E \quad (\forall x \in G), \tag{24}$$

then $T(x) = \lim\limits_{n\to\infty} \frac{f(3^n x)}{3^n}$ exists for all $x \in G$ and T is additive. Moreover,

$$T(x) - f(x) + f(\theta) = \eta(x) \quad (\forall x \in G). \tag{25}$$

Here

$$\begin{aligned}
\Psi(x) &\stackrel{\text{def}}{=} \tfrac{1}{78}\left[13\Psi_1(x) + 15\Psi_2(x) - 3\Psi_3(x) - 6\Psi_4(x) - 12\Psi_5(x)\right],\\
\Psi_1(x) &\stackrel{\text{def}}{=} \phi\left(x, -3x, x, -x, x, -x, x\right) + \phi\left(-x, 3x, -x, x, -x, -x, x\right)\\
&\quad -\phi\left(-x, 3x, -x, x, -x, x, -x\right) - \phi\left(x, -3x, x, -x, x, x, -x\right),\\
\Psi_2(x) &\stackrel{\text{def}}{=} \phi\left(x, -3x, x, -x, -x, x, x\right) + \phi\left(-x, 3x, -x, x, x, -x, -x\right)\\
&\quad -\phi\left(x, -3x, x, x, -x, -x, x\right) - \phi\left(-x, 3x, -x, -x, x, x, -x\right),\\
\Psi_3(x) &\stackrel{\text{def}}{=} \phi\left(x, -3x, -x, x, -x, x, x\right) + \phi\left(-x, 3x, x, -x, x, -x, -x\right)\\
&\quad -\phi\left(x, -3x, -x, x, x, -x, x\right) - \phi\left(-x, 3x, x, -x, -x, x, -x\right),\\
\Psi_4(x) &\stackrel{\text{def}}{=} \phi\left(x, x, x, x, -2x, x, -2x\right) + \phi\left(-x, -x, -x, -x, 2x, -x, 2x\right)\\
&\quad -\phi\left(x, x, x, -2x, x, x, -2x\right) - \phi\left(-x, -x, -x, 2x, -x, -x, 2x\right),
\end{aligned}$$

$$\begin{aligned}\Psi_5(x) \quad &\stackrel{\text{def}}{=} \phi\left(x, -\tfrac{x}{2}, \tfrac{x}{2}, -\tfrac{x}{2}, -\tfrac{x}{2}, \tfrac{x}{2}, -\tfrac{x}{2}, \right) + \phi\left(-x, \tfrac{x}{2}, -\tfrac{x}{2}, \tfrac{x}{2}, \tfrac{x}{2}, -\tfrac{x}{2}, \tfrac{x}{2}, \right)\\ &-\phi\left(x, -\tfrac{x}{2}, \tfrac{x}{2}, -\tfrac{x}{2}, \tfrac{x}{2}, -\tfrac{x}{2}, -\tfrac{x}{2}, \right) - \phi\left(-x, \tfrac{x}{2}, -\tfrac{x}{2}, \tfrac{x}{2}, -\tfrac{x}{2}, \tfrac{x}{2}, \tfrac{x}{2}, \right).\end{aligned}$$

Proof. It is enough to show the necessity. Define $g(x) = f(x) - f(\theta)$, then $g(\theta) = \theta$. The approximate remainders ϕ_g, ϕ_f of Eq(2) with respect to g, f respectively are equal. We represent them as ϕ. Let any $x \in G$. Putting $x_1 = x_3 = x_5 = x_7 = x$, $x_2 = -3x$, $x_4 = x_6 = -x$ in (3), we get

$$\begin{aligned}&7g(-\tfrac{x}{7}) - 6\left[4g\left(-\tfrac{x}{3}\right) + g\left(\tfrac{x}{3}\right)\right] + 5\left[2g\left(\tfrac{x}{5}\right) + g\left(-\tfrac{3x}{5}\right) + 4g\left(-\tfrac{x}{5}\right)\right]\\ &- \left[g\left(\tfrac{x}{2}\right) + 3g\left(-\tfrac{x}{2}\right)\right] + 3\left[3g\left(-\tfrac{x}{3}\right) + g\left(-x\right) + 3g\left(\tfrac{x}{3}\right)\right] - 2\left[2g(-x) + g(x)\right]\\ &= \phi\left(x, -3x, x, -x, x, -x, x\right).\end{aligned} \tag{26}$$

Setting $x_1 = x_3 = x_5 = x_6 = x$, $x_2 = -3x$, $x_4 = x_7 = -x$ in (3), we obtain

$$\begin{aligned}&7g(-\tfrac{x}{7}) - 6\left[4g\left(-\tfrac{x}{3}\right) + g\left(\tfrac{x}{3}\right)\right] + 5\left[2g\left(\tfrac{x}{5}\right) + g\left(-\tfrac{3x}{5}\right) + 4g\left(-\tfrac{x}{5}\right)\right]\\ &- \left[2g\left(\tfrac{x}{2}\right) + 4g\left(-\tfrac{x}{2}\right)\right] + 3\left[g\left(-\tfrac{x}{3}\right) + 2g\left(-x\right) + 4g\left(\tfrac{x}{3}\right)\right] - 2\left[2g(-x) + g(x)\right]\\ &= \phi\left(x, -3x, x, -x, x, x, -x\right).\end{aligned} \tag{27}$$

It follows from (26) and (27) that

$$\begin{aligned}&3\left[2g\left(-\tfrac{x}{3}\right) - g\left(\tfrac{x}{3}\right) - g\left(-x\right)\right] + \left[g\left(\tfrac{x}{2}\right) + g\left(-\tfrac{x}{2}\right)\right]\\ &= \phi\left(x, -3x, x, -x, x, -x, x\right) - \phi\left(x, -3x, x, -x, x, x, -x\right).\end{aligned} \tag{28}$$

With x by $-x$ in (28), we have

$$\begin{aligned}&3\left[2g\left(\tfrac{x}{3}\right) - g\left(-\tfrac{x}{3}\right) - g\left(x\right)\right] + \left[g\left(\tfrac{x}{2}\right) + g\left(-\tfrac{x}{2}\right)\right]\\ &= \phi\left(-x, 3x, -x, x, -x, x, -x\right) - \phi\left(-x, 3x, -x, x, -x, -x, x\right).\end{aligned} \tag{29}$$

It follows from (28) and (29) that

$$\begin{aligned}&3\left[\left(g(x) - 3g\left(\tfrac{x}{3}\right)\right) - \left(g(-x) - 3g\left(-\tfrac{x}{3}\right)\right)\right]\\ &= \phi\left(x, -3x, x, -x, x, -x, x\right) + \phi\left(-x, 3x, -x, x, -x, -x, x\right)\\ &-\phi\left(-x, 3x, -x, x, -x, x, -x\right) - \phi\left(x, -3x, x, -x, x, x, -x\right)\\ &\stackrel{\text{def}}{=} \Psi_1(x).\end{aligned} \tag{30}$$

Setting $x_1 = x_3 = x_6 = x_7 = x$, $x_2 = -3x$, $x_4 = x_5 = -x$ in (3), we get

$$\begin{aligned}
&7g(-\tfrac{x}{7}) - 6\left[4g\left(-\tfrac{x}{3}\right) + g\left(\tfrac{x}{3}\right)\right] + 5\left[3g\left(\tfrac{x}{5}\right) + 2g\left(-\tfrac{3x}{5}\right) + 2g\left(-\tfrac{x}{5}\right)\right]\\
&-\left[g\left(\tfrac{x}{2}\right) + g(-x) + g\left(-\tfrac{x}{2}\right)\right] + 3\left[4g\left(-\tfrac{x}{3}\right) + g\left(\tfrac{x}{3}\right) + g(x) + g(-x)\right]\\
&-2\left[3g(-x) + 2g(x)\right]\\
&= \phi\left(x, -3x, x, -x, -x, x, x\right).
\end{aligned} \tag{31}$$

Putting $x_1 = x_3 = x_4 = x_7 = x$, $x_2 = -3x$, $x_5 = x_6 = -x$ in (3), we obtain

$$\begin{aligned}
&7g(-\tfrac{x}{7}) - 6\left[4g\left(-\tfrac{x}{3}\right) + g\left(\tfrac{x}{3}\right)\right] + 5\left[3g\left(\tfrac{x}{5}\right) + 2g\left(-\tfrac{3x}{5}\right) + 2g\left(-\tfrac{x}{5}\right)\right]\\
&-2g\left(-\tfrac{x}{2}\right) + 3\left[5g\left(-\tfrac{x}{3}\right) + 2g\left(\tfrac{x}{3}\right)\right] - 2\left[3g(-x) + 2g(x)\right]\\
&= \phi\left(x, -3x, x, x, -x, -x, x\right).
\end{aligned} \tag{32}$$

It follows from (31) and (32) that

$$\begin{aligned}
&3\left[g(x) + g(-x) - g\left(\tfrac{x}{3}\right) - g\left(-\tfrac{x}{3}\right)\right] - \left[g(-x) + g\left(\tfrac{x}{2}\right) - g\left(-\tfrac{x}{2}\right)\right]\\
&= \phi\left(x, -3x, x, -x, -x, x, x\right) - \phi\left(x, -3x, x, x, -x, -x, x\right).
\end{aligned} \tag{33}$$

With x by $-x$ in (33), we have

$$\begin{aligned}
&3\left[g(-x) + g(x) - g\left(-\tfrac{x}{3}\right) - g\left(\tfrac{x}{3}\right)\right] - \left[g(x) + g\left(-\tfrac{x}{2}\right) - g\left(\tfrac{x}{2}\right)\right]\\
&= \phi\left(-x, 3x, -x, x, x, -x, -x\right) - \phi\left(-x, 3x, -x, -x, x, x, -x\right).
\end{aligned} \tag{34}$$

It follows from (33) and (34) that

$$\begin{aligned}
&2\left[\left(g(x) - 3g\left(\tfrac{x}{3}\right)\right) + \left(g(-x) - 3g\left(-\tfrac{x}{3}\right)\right)\right] + 3\left[g(x) + g(-x)\right]\\
&= \phi\left(x, -3x, x, -x, -x, x, x\right) + \phi\left(-x, 3x, -x, x, x, -x, -x\right)\\
&-\phi\left(x, -3x, x, x, -x, -x, x\right) - \phi\left(-x, 3x, -x, -x, x, x, -x\right)\\
&\stackrel{\text{def}}{=} \Psi_2(x).
\end{aligned} \tag{35}$$

Putting $x_1 = x_4 = x_6 = x_7 = x$, $x_2 = -3x$, $x_3 = x_5 = -x$ in (3), we obtain

$$\begin{aligned}&7g(-\tfrac{x}{7}) - 6\left[4g\left(-\tfrac{x}{3}\right) + g\left(\tfrac{x}{3}\right)\right] + 5\left[g\left(\tfrac{x}{5}\right) + g\left(\tfrac{3x}{5}\right) + 2g\left(-\tfrac{3x}{5}\right) + 3g\left(-\tfrac{x}{5}\right)\right]\\&- \left[2g\left(\tfrac{x}{2}\right) + 2g\left(-\tfrac{x}{2}\right) + g(-x)\right] + 3\left[2g\left(-x\right) + 2g\left(-\tfrac{x}{3}\right) + 2g\left(\tfrac{x}{3}\right) + g(x)\right]\\&-2\left[g(-x) + g(-2x) + 2g(x)\right]\\&= \phi\left(x, -3x, -x, x, -x, x, x\right).\end{aligned} \tag{36}$$

Letting $x_1 = x_4 = x_5 = x_7 = x$, $x_2 = -3x$, $x_3 = x_6 = -x$ in (3), we obtain

$$\begin{aligned}&7g(-\tfrac{x}{7}) - 6\left[4g\left(-\tfrac{x}{3}\right) + g\left(\tfrac{x}{3}\right)\right] + 5\left[g\left(\tfrac{x}{5}\right) + g\left(\tfrac{3x}{5}\right) + 2g\left(-\tfrac{3x}{5}\right) + 3g\left(-\tfrac{x}{5}\right)\right]\\&- \left[2g\left(\tfrac{x}{2}\right) + 4g\left(-\tfrac{x}{2}\right)\right] + 3\left[2g\left(-x\right) + g\left(-\tfrac{x}{3}\right) + 4g\left(\tfrac{x}{3}\right)\right]\\&-2\left[g(-x) + g(-2x) + 2g(x)\right]\\&= \phi\left(x, -3x, -x, x, x, -x, x\right).\end{aligned} \tag{37}$$

It follows from (36) and (37) that

$$\begin{aligned}&3\left[g(x) + g\left(-\tfrac{x}{3}\right) - 2g\left(\tfrac{x}{3}\right)\right] - \left[g(-x) - 2g\left(-\tfrac{x}{2}\right)\right]\\&= \phi\left(x, -3x, -x, x, -x, x, x\right) - \phi\left(x, -3x, -x, x, x, -x, x\right).\end{aligned} \tag{38}$$

With x by $-x$ in (38), we have

$$\begin{aligned}&3\left[g(-x) + g\left(\tfrac{x}{3}\right) - 2g\left(-\tfrac{x}{3}\right)\right] - \left[g(x) - 2g\left(\tfrac{x}{2}\right)\right]\\&= \phi\left(-x, 3x, x, -x, x, -x, -x\right) - \phi\left(-x, 3x, x, -x, -x, x, -x\right).\end{aligned} \tag{39}$$

It follows from (38) and (39) that

$$\begin{aligned}&2\left[g(x) + g(-x)\right] - 3\left[g\left(\tfrac{x}{3}\right) + g\left(-\tfrac{x}{3}\right)\right] + 2\left[g\left(\tfrac{x}{2}\right) + g\left(-\tfrac{x}{2}\right)\right]\\&= \phi\left(x, -3x, -x, x, -x, x, x\right) + \phi\left(-x, 3x, x, -x, x, -x, -x\right)\\&-\phi\left(x, -3x, -x, x, x, -x, x\right) - \phi\left(-x, 3x, x, -x, -x, x, -x\right)\\&\stackrel{\text{def}}{=} \Psi_3(x).\end{aligned} \tag{40}$$

Letting $x_1 = x_2 = x_3 = x_4 = x_6 = x$, $x_5 = x_7 == -2x$ in (3), we obtain

$$\begin{aligned}&7g(\tfrac{x}{7}) - 12g\left(\tfrac{x}{2}\right) + 5\left[3g\left(-\tfrac{x}{5}\right) + 4g\left(\tfrac{2x}{5}\right)\right] - \left[2g\left(-\tfrac{x}{2}\right) + g(x) + 4g\left(\tfrac{x}{4}\right)\right]\\&+3\left[2g(x) + g(-x)\right] - 2\left[3g(x) + 4g\left(-\tfrac{x}{2}\right)\right]\\&= \phi\left(x, x, x, x, -2x, x, -2x\right).\end{aligned} \tag{41}$$

Putting $x_1 = x_2 = x_3 = x_4 = x_6 = x$, $x_5 = x_7 == -2x$ in (3), we get

$$\begin{aligned}&7g(\tfrac{x}{7}) - 12g\left(\tfrac{x}{2}\right) + 5\left[3g\left(-\tfrac{x}{5}\right) + 4g\left(\tfrac{2x}{5}\right)\right] - \left[g\left(-\tfrac{x}{2}\right) + 6g\left(\tfrac{x}{4}\right)\right]\\&+3g(x) - 2\left[3g(x) + 4g\left(-\tfrac{x}{2}\right)\right]\\&= \phi\left(x, x, x, -2x, x, x, -2x\right).\end{aligned} \tag{42}$$

It follows from (41) and (42) that

$$\begin{aligned}&3\left[g(x) + g(-x)\right] - \left[g\left(-\tfrac{x}{2}\right) - 2g\left(\tfrac{x}{4}\right) + g(x)\right]\\&= \phi\left(x, x, x, x, -2x, x, -2x\right) - \phi\left(x, x, x, -2x, x, x, -2x\right).\end{aligned} \tag{43}$$

With $-x$ in place of x in (43), we obtain

$$\begin{aligned}&3\left[g(-x) + g(x)\right] - \left[g\left(\tfrac{x}{2}\right) - 2g\left(-\tfrac{x}{4}\right) + g(-x)\right]\\&= \phi\left(-x, -x, -x, -x, 2x, -x, 2x\right) - \phi\left(-x, -x, -x, 2x, -x, -x, 2x\right).\end{aligned} \tag{44}$$

It follows from (43) and (44) that

$$\begin{aligned}&5\left[g(-x) + g(x)\right] - \left[g\left(\tfrac{x}{2}\right) + g\left(-\tfrac{x}{2}\right)\right] + 2\left[g\left(\tfrac{x}{4}\right) + g\left(-\tfrac{x}{4}\right)\right]\\&= \phi\left(x, x, x, x, -2x, x, -2x\right) + \phi\left(-x, -x, -x, -x, 2x, -x, 2x\right)\\&-\phi\left(x, x, x, -2x, x, x, -2x\right) - \phi\left(-x, -x, -x, 2x, -x, -x, 2x\right)\\&\stackrel{\text{def}}{=} \Psi_4(x).\end{aligned} \tag{45}$$

Letting $x_1 = 2x$, $x_2 = x_4 = x_5 = x_7 = -x$, $x_3 == x_6 = x$ in (3), we get

$$\begin{aligned}&-6\left[4g\left(\tfrac{x}{6}\right) + 2g\left(-\tfrac{x}{6}\right) + g\left(-\tfrac{x}{3}\right)\right] + 5\left[2g\left(-\tfrac{x}{5}\right) + g\left(\tfrac{2x}{5}\right)\right]\\&- \left[2g\left(-\tfrac{x}{2}\right) + 4g\left(\tfrac{x}{4}\right)\right] + 3\left[2g\left(\tfrac{2x}{3}\right) + 4g\left(-\tfrac{x}{3}\right)\right]\\&-2\left[2g\left(\tfrac{x}{2}\right) + g(-x)\right]\\&= \phi(2x, -x, x, -x, -x, x, -x).\end{aligned} \tag{46}$$

Putting $x_1 = 2x$, $x_2 = x_4 = x_6 = x_7 = -x$, $x_3 == x_5 = x$ in (3), we obtain

$$\begin{aligned}
&-6\left[4g\left(\tfrac{x}{6}\right)+2g\left(-\tfrac{x}{6}\right)+g\left(-\tfrac{x}{3}\right)\right]+5\left[2g\left(-\tfrac{x}{5}\right)+g\left(\tfrac{2x}{5}\right)\right]\\
&-\left[g\left(-\tfrac{x}{2}\right)+3g\left(\tfrac{x}{4}\right)+g\left(-\tfrac{x}{4}\right)\right]+3\left[g\left(\tfrac{2x}{3}\right)+3g\left(-\tfrac{x}{3}\right)+g\left(\tfrac{x}{3}\right)\right]\\
&-2\left[2g\left(\tfrac{x}{2}\right)+g(-x)\right]\\
&=\phi(2x,-x,x,-x,x,-x,-x).
\end{aligned} \tag{47}$$

It follows from (46) and (47) that

$$\begin{aligned}
&3\left[g\left(\tfrac{2x}{3}\right)+g\left(-\tfrac{x}{3}\right)-g\left(\tfrac{x}{3}\right)\right]-\left[g\left(-\tfrac{x}{2}\right)+g\left(\tfrac{x}{4}\right)-g\left(-\tfrac{x}{4}\right)\right]\\
&=\phi(2x,-x,x,-x,-x,x,-x)-\phi(2x,-x,x,-x,x,-x,-x).
\end{aligned} \tag{48}$$

With x by $-x$ in (48), we get

$$\begin{aligned}
&3\left[g\left(-\tfrac{2x}{3}\right)+g\left(\tfrac{x}{3}\right)-g\left(-\tfrac{x}{3}\right)\right]-\left[g\left(\tfrac{x}{2}\right)+g\left(-\tfrac{x}{4}\right)-g\left(\tfrac{x}{4}\right)\right]\\
&=\phi(-2x,x,-x,x,x,-x,x)-\phi(-2x,x,-x,x,-x,x,x).
\end{aligned} \tag{49}$$

It follows from (48) and (49) that

$$\begin{aligned}
&3\left[g\left(\frac{2x}{3}\right)+g\left(-\frac{2x}{3}\right)\right]-\left[g\left(\frac{x}{2}\right)+T\left(-\frac{x}{2}\right)\right]\\
&=\phi(2x,-x,x,-x,-x,x,-x)+\phi(-2x,x,-x,x,x,-x,x)\\
&-\phi(2x,-x,x,-x,x,-x,-x)-\phi(-2x,x,-x,x,-x,x,x).
\end{aligned}$$

With $\frac{x}{2}$ in place of x in the last equation, we conclude

$$\begin{aligned}
&3\left[g\left(\tfrac{x}{3}\right)+g\left(-\tfrac{x}{3}\right)\right]-\left[g\left(\tfrac{x}{4}\right)+g\left(-\tfrac{x}{4}\right)\right]\\
&=\phi\left(x,-\tfrac{x}{2},\tfrac{x}{2},-\tfrac{x}{2},-\tfrac{x}{2},\tfrac{x}{2},-\tfrac{x}{2},\right)+\phi\left(-x,\tfrac{x}{2},-\tfrac{x}{2},\tfrac{x}{2},\tfrac{x}{2},-\tfrac{x}{2},\tfrac{x}{2},\right)\\
&-\phi\left(x,-\tfrac{x}{2},\tfrac{x}{2},-\tfrac{x}{2},\tfrac{x}{2},-\tfrac{x}{2},-\tfrac{x}{2},\right)-\phi\left(-x,\tfrac{x}{2},-\tfrac{x}{2},\tfrac{x}{2},-\tfrac{x}{2},\tfrac{x}{2},\tfrac{x}{2},\right)\\
&\stackrel{\text{def}}{=}\Psi_5(x).
\end{aligned} \tag{50}$$

It follows from (45) and (50) that

$$5\left[g(x)+g(-x)\right]+6\left[g\left(\frac{x}{3}\right)+g\left(-\frac{x}{3}\right)\right]-\left[g\left(\frac{x}{2}\right)+g\left(-\frac{x}{2}\right)\right]=\Psi_4(x)+2\Psi_5(x). \tag{51}$$

It follows from (40) and (51) that

$$12\left[g(x)+g(-x)\right]+9\left[g\left(\tfrac{x}{3}\right)+g\left(-\tfrac{x}{3}\right)\right]=\Psi_3(x)+2\Psi_4(x)+4\Psi_5(x)$$

i.e.,

$$\begin{aligned}&-3\left[\left(g(x)-3g\left(\tfrac{x}{3}\right)\right)+\left(g(-x)\ -\ 3g\left(-\tfrac{x}{3}\right)\right)\right]+15\left[g(x)+g(-x)\right]\\&=\Psi_3(x)+2\Psi_4(x)+4\Psi_5(x)\end{aligned} \tag{52}$$

It follows from (35) and (52) that

$$13\left[\left(g(x)-3g\left(\frac{x}{3}\right)\right)+\left(g(-x)\ -\ 3g\left(-\frac{x}{3}\right)\right)\right]=5\Psi_2(x)-\Psi_3(x)-2\Psi_4(x)-4\Psi_5(x). \tag{53}$$

It follows from (53) and (30) that

$$2\left[g(x)-3g\left(\frac{x}{3}\right)\right]=\frac{1}{3}\Psi_1(x)+\frac{1}{13}\left[5\Psi_2(x)-\Psi_3(x)-2\Psi_4(x)-4\Psi_5(x)\right]$$

i.e.,

$$g(x)-3g\left(\frac{x}{3}\right)=\frac{1}{78}\left[13\Psi_1(x)+15\Psi_2(x)-3\Psi_3(x)-6\Psi_4(x)-12\Psi_5(x)\right]\overset{\text{def}}{=}\Psi(x).$$

With x by $3x$ and divided by 3 the last equality turns to

$$\frac{1}{3}g(3x)-g(x)=\frac{1}{3}\Psi(3x). \tag{54}$$

We shall use induction on n to prove

$$\frac{1}{3^n}g(3^n x)-g(x)=\sum_{k=1}^{n}\frac{\Psi(3^k x)}{3^k}\quad(\forall n\in\mathbf{N}). \tag{55}$$

Firstly, (55) is true for $n=1$, since (54). Secondly, we make the induction hypothesis that (55) holds for a certain $n=m-1$. Then we have

$$\begin{aligned}\frac{1}{3^m}g(3^m x)-g(x)&=\frac{1}{3}\left[\frac{1}{3^{m-1}}g(3^{m-1}(3x))-g(3x)\right]+\frac{1}{3}g(3x)-g(x)\\&=\frac{1}{3}\sum_{k=1}^{m-1}\frac{\Psi(3^k(3x))}{3^k}+\frac{1}{3}\Psi(3x)=\sum_{k=1}^{m}\frac{\Psi(3^k x)}{3^k},\end{aligned}$$

which completes the induction proof. Define $T(x)=\lim\limits_{n\to\infty}\frac{g(3^n x)}{3^n}$. Obviously, $T(x)=\lim\limits_{n\to\infty}\frac{f(3^n x)}{3^n}$. Combining (24) and (55), we have that $T(x)$ exists and

$$T(x)-g(x)=\eta(x).$$

Substituting the definition of g into the last equality gives birth to

$$T(x)-f(x)+f(\theta)=\eta(x).$$

Finally, we verify the additivity of T. By the definition of T, we get

$$T(\theta) = \lim_{n\to\infty} \frac{g(3^n\theta)}{3^n} = \theta.$$

So T is a solution of Eq.(2) in view of (23). Therefore $T(x) = T^*(x) + T(\theta) = T^*(x)$ by virtue of Theorem 1, where T^* is additive. We prove that T is additive. □

Applying Theorem 2, we shall derive the Hyers-Ulam stability of Eq.(2) which is closely connected with the β-homogeneity of the norm on F^*-spaces. At the same time, we allow p_i $(i = 1, 2, \cdots, 7)$ to be different.

Let X be a linear space. A non-negative valued function $\|\cdot\|$ defined on X is called an F-norm if it satisfies the following conditions:

(n1) $\|x\| = 0$ if and only if $x = 0$;
(n2) $\|ax\| = \|x\|$ for all a, $|a| = 1$;
(n3) $\|x + y\| \le \|x\| + \|y\|$;
(n4) $\|a_n x\| \longrightarrow 0$ provided $a_n \longrightarrow 0$;
(n5) $\|a x_n\| \longrightarrow 0$ provided $x_n \longrightarrow 0$.

A space X with an F-norm is called an F^*-space. An F-pseudonorm ($\|x\| = 0$ does not necessarily imply that $x = 0$ in (n1)) is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{R}$. A complete F^*-space is said to be an F-space.

Corollary 1. *Suppose that G is an F^*-space and E a $\beta-$homogeneous F-space $(0 < \beta \le 1)$. Given $\varepsilon_i, \delta \ge 0$ and $p_i < \beta$, if ϕ satisfies*

$$\|\phi(x_1, x_2, \cdots, x_7)\| \le \delta + \varepsilon_1\|x_1\|^{p_1} + \varepsilon_2\|x_2\|^{p_2} + \cdots + \varepsilon_7\|x_7\|^{p_7} \ (\forall x_i \in G), \quad (56)$$

then there exists a unique additive mapping $T\colon G \to E$ such that

$$\|T(x) - f(x) + f(\theta)\| \le A\delta + \varepsilon_1 B_1\|x\|^{p_1} + \varepsilon_2 B_2\|x\|^{p_2} + \cdots + \varepsilon_7 B_7\|x\|^{p_7}$$

for all $x \in G$, where

$$A \stackrel{\text{def}}{=} \frac{4}{78^\beta(3^\beta - 1)}\left(13^\beta + 15^\beta + 3^\beta + 6^\beta + 12^\beta\right),$$

$$B_1 \stackrel{\text{def}}{=} \frac{4 \times 3^{p_1}}{78^\beta(3^\beta - 3^{p_1})}\left(13^\beta + 15^\beta + 3^\beta + 6^\beta + 12^\beta\right),$$

$$B_2 \stackrel{\text{def}}{=} \frac{4 \times 3^{p_2}}{78^\beta(3^\beta - 3^{p_2})}\left[3^{p_2} \times \left(13^\beta + 15^\beta + 3^\beta\right) + 6^\beta + \frac{12^\beta}{2^{p_2}}\right],$$

$$B_3 \stackrel{\text{def}}{=} \frac{4 \times 3^{p_3}}{78^\beta(3^\beta - 3^{p_3})}\left(13^\beta + 15^\beta + 3^\beta + 6^\beta + \frac{12^\beta}{2^{p_3}}\right),$$

$$B_4 \overset{\text{def}}{=} \frac{2\times 3^{p_4}}{78^\beta(3^\beta-3^{p_4})}\left[2\times\left(13^\beta+15^\beta+3^\beta+\frac{12^\beta}{2^{p_4}}\right)+\left(2^{p_4}+1\right)\times 6^\beta\right],$$
$$B_5 \overset{\text{def}}{=} \frac{2\times 3^{p_5}}{78^\beta(3^\beta-3^{p_5})}\left[2\times\left(13^\beta+15^\beta+3^\beta+\frac{12^\beta}{2^{p_5}}\right)+\left(2^{p_5}+1\right)\times 6^\beta\right],$$
$$B_6 \overset{\text{def}}{=} \frac{4\times 3^{p_6}}{78^\beta(3^\beta-3^{p_6})}\left(13^\beta+15^\beta+3^\beta+6^\beta+\frac{12^\beta}{2^{p_6}}\right),$$
$$B_7 \overset{\text{def}}{=} \frac{4\times 3^{p_7}}{78^\beta(3^\beta-3^{p_7})}\left(13^\beta+15^\beta+3^\beta+6^\beta\times 2^{p_7}+\frac{12^\beta}{2^{p_7}}\right).$$

When some p_i is strictly less than 0, we may suppose that (56) holds for $x_i \neq \theta$. And the domain of T is $G\backslash\{\theta\}$ instead of G. Subscript i of this corollary is a positive integer from 1 to 7.

If $f(x)$ is continuous in G, then $T(x)$ is linear.

Proof. Define g as above. Let any $x_i \in G$ ($i = 1, 2, \cdots, 7$).

Firstly, we prove the existence of T. According to (56), we get that

$$\|\phi\left(3^n x_1, 3^n x_2, \cdots, 3^n x_7\right)\| \le \delta+\varepsilon_1 3^{np_1}\|x_1\|^{p_1}+\varepsilon_2 3^{np_2}\|x_2\|^{p_2}+\cdots+\varepsilon_7 3^{np_7}\|x_7\|^{p_7}.$$

Since $0 \le p_i < \beta$ ($i = 1, 2, \cdots, 7$), we have

$$\lim_{n\to\infty}\|\frac{\phi\left(3^n x_1, 3^n x_2, \cdots, 3^n x_7\right)}{3^n}\| \le \lim_{n\to\infty}\left[\frac{\delta}{3^{n\beta}}+\sum_{i=1}^{7}\frac{1}{3^{n(\beta-p_i)}}\varepsilon_i\|x_i\|^{p_i}\right]=0.$$

Furthermore, by the method similar to the proof of Theorem 2, we infer from (55) that

$$\frac{1}{3^n}g(3^n x)-g(x)=\sum_{k=1}^{n}\frac{\Psi(3^k x)}{3^k}$$

holds for any $n \in \mathbb{N}$, where Ψ is as above. It is clear that $\sum_{k=1}^{\infty}\frac{\Psi(3^k x)}{3^k}$ exits for every $x \in G$. Indeed, from the above, we can get

$$\begin{aligned}\frac{g(3^m x)}{3^m}-\frac{g(3^n x)}{3^n} &= \frac{1}{3^n}\left[\frac{g\left(3^{m-n}(3^n x)\right)}{3^{m-n}}-f(3^n x)\right]\\ &= \frac{1}{3^n}\sum_{k=1}^{m-n}\frac{\Psi(3^{n+k}x)}{3^k}=\sum_{k=n+1}^{m}\frac{\Psi(3^k x)}{3^k}.\end{aligned}$$

Combining (56) and β-homogeneity of the norm in E, we have

$$\begin{aligned}
&\|\Psi_1(3^k x)\| \\
&= \|\phi\left(3^k x, -3^{k+1}x, 3^k x, -3^k x, 3^k x, -3^k x, 3^k x\right) \\
&+\phi\left(-3^k x, 3^{k+1}x, -3^k x, 3^k x, -3^k x, -3^k x, 3^k x\right) \\
&-\phi\left(-3^k x, 3^{k+1}x, -3^k x, 3^k x, -3^k x, 3^k x, -3^k x\right) \\
&-\phi\left(3^k x, -3^{k+1}x, 3^k x, -3^k x, 3^k x, 3^k x, -3^k x\right)\| \\
&\leq \|\phi\left(3^k x, -3^{k+1}x, 3^k x, -3^k x, 3^k x, -3^k x, 3^k x\right)\| \\
&+\|\phi\left(-3^k x, 3^{k+1}x, -3^k x, 3^k x, -3^k x, -3^k x, 3^k x\right)\| \\
&+\|\phi\left(-3^k x, 3^{k+1}x, -3^k x, 3^k x, -3^k x, 3^k x, -3^k x\right)\| \\
&+\|\phi\left(3^k x, -3^{k+1}x, 3^k x, -3^k x, 3^k x, 3^k x, -3^k x\right)\| \\
&\leq 4\left[\delta + \varepsilon_1 3^{kp_1}\|x\|^{p_1} + \varepsilon_2 3^{(k+1)p_2}\|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{kp_i}\|x\|^{p_i}\right].
\end{aligned}$$

As in the above proof, we can get

$$\begin{aligned}
\|\Psi_2(3^k x)\| &\leq 4\left[\delta + \varepsilon_1 3^{kp_1}\|x\|^{p_1} + \varepsilon_2 3^{(k+1)p_2}\|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{kp_i}\|x\|^{p_i}\right], \\
\|\Psi_3(3^k x)\| &\leq 4\left[\delta + \varepsilon_1 3^{kp_1}\|x\|^{p_1} + \varepsilon_2 3^{(k+1)p_2}\|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{kp_i}\|x\|^{p_i}\right], \\
\|\Psi_4(3^k x)\| &\leq 4\left[\delta + \sum_{i=1}^{3} \varepsilon_i 3^{kp_i}\|x\|^{p_i} + \varepsilon_6 3^{kp_6}\|x\|^{p_6} + \varepsilon_7 2^{p_7} \times 3^{kp_7}\|x\|^{p_7}\right] \\
&\quad +2\sum_{i=4}^{5} \varepsilon_i \left(2^{p_i}+1\right) \times 3^{kp_i}\|x\|^{p_i}, \\
\|\Psi_5(3^k x)\| &\leq 4\left[\delta + \varepsilon_1 3^{kp_1}\|x\|^{p_1} + \sum_{i=2}^{7} \varepsilon_i 3^{kp_i} \times \frac{1}{2^{p_i}}\|x\|^{p_i}\right].
\end{aligned}$$

Furthermore, for any $m > n$, $m, n \in \mathbb{N}$, we have

$$\left\|\frac{g(3^m x)}{3^m} - \frac{g(3^n x)}{3^n}\right\| = \left\|\sum_{k=n+1}^{m} \frac{\Psi(3^k x)}{3^k}\right\| \leq \sum_{k=n+1}^{m} \left\|\frac{\Psi(3^k x)}{3^k}\right\|$$

$$
\begin{aligned}
&= \sum_{k=n+1}^{m} \left\| \frac{1}{3^k \times 78} \left[13\Psi_1(3^k x) + 15\Psi_2(3^k x) - 3\Psi_3(3^k x) - 6\Psi_4(x) - 12\Psi_5(3^k x) \right] \right\| \\
&\leq \sum_{k=n+1}^{m} \frac{1}{3^{k\beta_2} 78^{\beta_2}} \left[13^{\beta_2} \|\Psi_1(3^k x)\| + 15^{\beta_2} \|\Psi_2(3^k x)\| + 3^{\beta_2} \|\Psi_3(3^k x)\| \right] \\
&+ \sum_{k=n+1}^{m} \frac{1}{3^{k\beta_2} 78^{\beta_2}} \left[6^{\beta_2} \|\Psi_4(3^k x)\| + 12^{\beta_2} \|\Psi_5(3^k x)\| \right] \\
&\leq \sum_{k=n+1}^{m} \frac{4 \times 13^{\beta_2}}{3^{k\beta_2} 78^{\beta_2}} \left[\varepsilon_1 3^{kp_1\beta_1} \|x\|^{p_1} + \varepsilon_2 3^{(k+1)p_2\beta_1} \|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{kp_i\beta_1} \|x\|^{p_i} \right] \\
&+ \sum_{k=n+1}^{m} \frac{4 \times 15^{\beta_2}}{3^{k\beta_2} 78^{\beta_2}} \left[\varepsilon_1 3^{kp_1\beta_1} \|x\|^{p_1} + \varepsilon_2 3^{(k+1)p_2\beta_1} \|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{kp_i\beta_1} \|x\|^{p_i} \right] \\
&+ \sum_{k=n+1}^{m} \frac{4 \times 3^{\beta_2}}{3^{k\beta_2} 78^{\beta_2}} \left[\varepsilon_1 3^{kp_1\beta_1} \|x\|^{p_1} + \varepsilon_2 3^{(k+1)p_2\beta_1} \|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{kp_i\beta_1} \|x\|^{p_i} \right] \\
&+ \sum_{k=n+1}^{m} \frac{4 \times 6^{\beta_2}}{3^{k\beta_2} 78^{\beta_2}} \left[\sum_{i=1}^{3} \varepsilon_i 3^{kp_i\beta_1} \|x\|^{p_i} + \varepsilon_6 3^{kp_6\beta_1} \|x\|^{p_6} + \varepsilon_7 2^{p_7\beta_1} \times 3^{kp_7\beta_1} \|x\|^{p_7} \right] \\
&+ \sum_{k=n+1}^{m} \frac{2 \times 6^{\beta_2}}{3^{k\beta_2} 78^{\beta_2}} \left[\sum_{i=4}^{5} \varepsilon_i \left(2^{p_i\beta_1} + 1 \right) \times 3^{kp_i\beta_1} \|x\|^{p_i} \right] \\
&+ \sum_{k=n+1}^{m} \frac{4 \times 12^{\beta_2}}{3^{k\beta_2} 78^{\beta_2}} \left[\varepsilon_1 3^{kp_1\beta_1} \|x\|^{p_1} + \sum_{i=2}^{7} \varepsilon_i 3^{kp_i\beta_1} \times \frac{1}{2^{p_i\beta_1}} \|x\|^{p_i} \right] \\
&= \sum_{k=n+1}^{m} \frac{4}{3^{k(\beta_2 - p_1\beta_1)} 78^{\beta_2}} \times \left[13^{\beta_2} + 15^{\beta_2} + 3^{\beta_2} + 6^{\beta_2} + 12^{\beta_2} \right] \varepsilon_1 \times \|x\|^{p_1} \\
&+ \sum_{k=n+1}^{m} \frac{4\varepsilon_2 \|x\|^{p_2}}{3^{k(\beta_2 - p_2\beta_1)} 78^{\beta_2}} \left[\left(13^{\beta_2} + 15^{\beta_2} + 3^{\beta_2} \right) 3^{p_2\beta_1} + 6^{\beta_2} + \frac{12^{\beta_2}}{2^{p_2\beta_1}} \right] \\
&+ \sum_{k=n+1}^{m} \frac{4}{3^{k(\beta_2 - p_3\beta_1)} 78^{\beta_2}} \left[13^{\beta_2} + 15^{\beta_2} + 3^{\beta_2} + 6^{\beta_2} + \frac{12^{\beta_2}}{2^{p_3\beta_1}} \right] \times \varepsilon_3 \times \|x\|^{p_3} \\
&+ \sum_{k=n+1}^{m} \frac{\varepsilon_4 \|x\|^{p_4}}{3^{k(\beta_2 - p_4\beta_1)} 78^{\beta_2}} \left[4 \left(13^{\beta_2} + 15^{\beta_2} + 3^{\beta_2} + \frac{12^{\beta_2}}{2^{p_4\beta_1}} \right) + 2(2^{p_4\beta_1} + 1) 6^{\beta_2} \right] \\
&+ \sum_{k=n+1}^{m} \frac{\varepsilon_5 \|x\|^{p_5}}{3^{k(\beta_2 - p_5\beta_1)} 78^{\beta_2}} \left[4 \left(13^{\beta_2} + 15^{\beta_2} + 3^{\beta_2} + \frac{12^{\beta_2}}{2^{p_5\beta_1}} \right) + 2(2^{p_5\beta_1} + 1) 6^{\beta_2} \right]
\end{aligned}
$$

$$+\sum_{k=n+1}^{m}\frac{4\varepsilon_6\|x\|^{p_6}}{3^{k(\beta_2-p_6\beta_1)}78^{\beta_2}}\left[13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+6^{\beta_2}+\frac{12^{\beta_2}}{2^{p_6\beta_1}}\right]$$

$$+\sum_{k=n+1}^{m}\frac{4\times\varepsilon_7\|x\|^{p_7}}{3^{k(\beta_2-p_7\beta_1)}78^{\beta_2}}\left[13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+6^{\beta_2}2^{p_7}+\frac{12^{\beta_2}}{2^{p_7\beta_1}}\right].$$

Hence, $\left\{\frac{g(3^n x)}{3^n}\right\}$ is a Cauchy sequence of E because of $p_i<\beta$ $(i=1,2,\cdots,7)$. It converges to an element of E since E is complete. From (55), we infer

$$\sum_{k=1}^{\infty}\frac{\Psi(3^k x)}{3^k}$$

exists for every $x\in G$. Thus, by Theorem 2, $T(x)=\lim_{n\to\infty}\frac{g(3^n x)}{3^n}=\lim_{n\to\infty}\frac{g(3^n x)+f(\theta)}{3^n}$ $=\lim_{n\to\infty}\frac{f(3^n x)}{3^n}$ and it is additive. In addition, $T(x)-f(x)+f(\theta)=\sum_{k=1}^{\infty}\frac{\Psi(3^k x)}{3^k}$ follows from (55). Furthermore,

$$\begin{aligned}
&\|T(x)-f(x)+f(\theta)\|\\
&=\|\sum_{k=1}^{\infty}\frac{\Psi(3^k x)}{3^k}\|\\
&\le\sum_{k=1}^{\infty}\frac{4}{3^{k\beta}\times 78^{\beta}}\times\left[13^{\beta}+15^{\beta}+3^{\beta}+6^{\beta}+12^{\beta}\right]\times\delta\\
&+\sum_{k=1}^{\infty}\frac{4}{3^{k(\beta-p_1)}\times 78^{\beta}}\times\left[13^{\beta}+15^{\beta}+3^{\beta}+6^{\beta}+12^{\beta}\right]\times\varepsilon_1\times\|x\|^{p_1}\\
&+\sum_{k=1}^{\infty}\frac{4}{3^{k(\beta-p_2)}\times 78^{\beta}}\times\left[\left(13^{\beta}+15^{\beta}+3^{\beta}\right)\times 3^{p_2}+6^{\beta}+\frac{12^{\beta}}{2^{p_2}}\right]\times\varepsilon_2\times\|x\|^{p_2}\\
&+\sum_{k=1}^{\infty}\frac{4}{3^{k(\beta-p_3)}\times 78^{\beta}}\times\left[13^{\beta}+15^{\beta}+3^{\beta}+6^{\beta}+\frac{12^{\beta}}{2^{p_3}}\right]\times\varepsilon_3\times\|x\|^{p_3}\\
&+\sum_{k=1}^{\infty}\frac{1}{3^{k(\beta-p_4)}\times 78^{\beta}}\times\left[4\left(13^{\beta}+15^{\beta}+3^{\beta}+\frac{12^{\beta}}{2^{p_4}}\right)+2(2^{p_4}+1)\times 6^{\beta}\right]\varepsilon_4\|x\|^{p_4}\\
&+\sum_{k=1}^{\infty}\frac{1}{3^{k(\beta-p_5)}\times 78^{\beta}}\times\left[4\left(13^{\beta}+15^{\beta}+3^{\beta}+\frac{12^{\beta}}{2^{p_5}}\right)+2(2^{p_5}+1)\times 6^{\beta}\right]\varepsilon_5\|x\|^{p_5}\\
&+\sum_{k=1}^{\infty}\frac{4}{3^{k(\beta-p_6)}\times 78^{\beta}}\times\left[13^{\beta}+15^{\beta}+3^{\beta}+6^{\beta}+\frac{12^{\beta}}{2^{p_6}}\right]\times\varepsilon_6\times\|x\|^{p_6}
\end{aligned}$$

$$+\sum_{k=1}^{\infty}\frac{4}{3^{k(\beta-p_7)}\times 78^{\beta}}\times\left[13^{\beta}+15^{\beta}+3^{\beta}+6^{\beta}\times 2^{p_7}+\frac{12^{\beta}}{2^{p_7}}\right]\times\varepsilon_7\times\|x\|^{p_7}$$
$$=A\delta+\varepsilon_1B_1\|x\|^{p_1}+\varepsilon_2B_2\|x\|^{p_2}+\cdots+\varepsilon_7B_7\|x\|^{p_7}$$

for all $x\in G$.

Secondly, we prove the uniqueness of T. If $U\colon G\to E$ is another additive mapping satisfying

$$\|U(x)-f(x)+f(\theta)\|\le A\delta+\varepsilon_1B_1\|x\|^{p_1}+\varepsilon_2B_2\|x\|^{p_2}+\cdots+\varepsilon_7B_7\|x\|^{p_7}$$

for all $x\in G$, it follows from the last two inequalities that

$$\begin{aligned}\|U(x)-T(x)\|&=\frac{1}{n^{\beta}}\|U(nx)-T(nx)\|\\&=\frac{1}{n^{\beta}}\|U(nx)-f(nx)+f(\theta)-T(nx)+f(nx)-f(\theta)\|\\&\le\frac{1}{n^{\beta}}[\|U(nx)-f(nx)+f(\theta)\|+\|T(nx)-f(nx)+f(\theta)\|]\\&\le\frac{2}{n^{\beta}}(A\delta+\varepsilon_1B_1\|nx\|^{p_1}+\varepsilon_2B_2\|nx\|^{p_2}+\cdots+\varepsilon_7B_7\|nx\|^{p_7})\\&=2\left[\frac{A\delta}{n^{\beta}}+\frac{\varepsilon_1B_1}{n^{\beta-p_1}}\|x\|^{p_1}+\frac{\varepsilon_2B_2}{n^{\beta-p_2}}\|x\|^{p_2}+\cdots+\frac{\varepsilon_7B_7}{n^{\beta-p_7}}\|x\|^{p_7}\right].\end{aligned}$$

Taking the limit as $n\to\infty$, we get $\|U(x)-T(x)\|\to 0$ since $p_i<\beta$ $(i=1,2,\cdots,7)$. As a result, $U(x)=T(x)$ for all $x\in G$.

Finally, we consider the linearity of T. If $f(x)$ is continuous in G, then $T_n(x)=\frac{f(3^nx)}{3^n}$ is continuous in G for any $n\in\mathbb{N}$. It can be seen that $\{T_n(x)\}$ uniformly converges to $T(x)$ in each closed ball from the above proof. So $T(x)$ is continuous. It concludes $T(x)$ is linear. Hence, the result holds. □

Theorem 3. *The approximate remainder ϕ satisfies*

$$\lim_{n\to\infty}3^n\phi(3^{-n}x_1,3^{-n}x_2,\cdots,3^{-n}x_7)=\theta\quad(\forall x_i\in G,i=1,2,\cdots,7)\tag{57}$$

$$\sum_{k=1}^{\infty}3^{k-1}\Psi(3^{-(k-1)}x)=\eta(x)\in E\quad(\forall x\in G)\tag{58}$$

if and only if the limit $T(x)=\lim\limits_{n\to\infty}3^n[f(3^{-n}x)-f(\theta)]$ exists for all $x\in G$ and T is additive, where Ψ is as above. Moreover,

$$T(x)-f(x)+f(\theta)=\eta(x)\quad(\forall x\in G).\tag{59}$$

Proof. Let $g(x) = f(x) - f(\theta)$. Note that by virtue of (54) we conclude by induction that

$$g(x) - 3^n g(3^{-n}x) = \sum_{k=1}^{n} 3^{k-1}\Psi(3^{-(k-1)}x) \quad (\forall x \in G \text{ and } n \in \mathbb{N}).$$

□

Corollary 2. *Suppose that G is a β-homogeneous F^*-space $(0 < \beta \le 1)$ and E an F-space. Given $\varepsilon_i \in [0, +\infty)$ $(i = 1, 2, \cdots, 7)$ and $p_i \in (\frac{1}{\beta}, +\infty)$ $(i = 1, 2, \cdots, 7)$, if ϕ satisfies*

$$\|\phi(x_1, x_2, \cdots, x_7)\| \le \varepsilon_1\|x_1\|^{p_1} + \varepsilon_2\|x_2\|^{p_2} + \cdots + \varepsilon_7\|x_7\|^{p_7} \ (\forall x_i \in G) \quad (60)$$

then there exists a unique additive mapping $T\colon G \to E$ such that

$$\|T(x) - f(x) + f(\theta)\| \le \varepsilon_1 B_1\|x\|^{p_1} + \varepsilon_2 B_2\|x\|^{p_2} + \cdots + \varepsilon_7 B_7\|x\|^{p_7} \quad (\forall x \in G)$$

where

$$B_1 \overset{\text{def}}{=} \frac{20\times 3^{\beta p_1}}{3^{\beta p_1} - 3}, \qquad B_2 \overset{\text{def}}{=} \frac{4\times 3^{\beta p_2}}{3^{\beta p_2} - 3}\left(3^{\beta p_2 + 1} + 1 + \frac{1}{2^{\beta p_2}}\right),$$
$$B_3 \overset{\text{def}}{=} \frac{4\times 3^{\beta p_3}}{3^{\beta p_3} - 3}\left(4 + \frac{1}{2^{\beta p_3}}\right), \qquad B_4 \overset{\text{def}}{=} \frac{2\times 3^{\beta p_4}}{3^{\beta p_4} - 3}\left(7 + 2^{\beta p_4} + \frac{2}{2^{\beta p_4}}\right),$$
$$B_5 \overset{\text{def}}{=} \frac{2\times 3^{\beta p_5}}{3^{\beta p_5} - 3}\left(7 + 2^{\beta p_5} + \frac{2}{2^{\beta p_5}}\right), \quad B_6 \overset{\text{def}}{=} \frac{4\times 3^{\beta p_6}}{3^{\beta p_6} - 3}\left(4 + \frac{1}{2^{\beta p_6}}\right),$$
$$B_7 \overset{\text{def}}{=} \frac{4\times 3^{\beta p_7}}{3^{\beta p_7} - 3}\left(3 + 2^{\beta p_7} + \frac{1}{2^{\beta p_7}}\right).$$

Especially, $T(x)$ is linear under the condition that $f(x)$ is continuous.

Proof. Existence: Let $g(x) = f(x) - f(\theta)$ for any $x \in G$. We may assume that F-norm $\|\cdot\|$ is non-decreasing. If not, by [8, Theorem 1.2.2], there exists a norm $|||\cdot|||$ equivalent to the original norm $\|\cdot\|$(i.e. $\exists\ \lambda,\ \mu$ such that $\lambda|||\cdot||| \le \|\cdot\| \le \mu|||\cdot|||$) and it is non-decreasing. Furthermore we choose the above μ which is as small as possible. Then

$$\begin{aligned}\|\phi(x_1, x_2, \cdots, x_7)\| &\le \varepsilon_1\|x_1\|^{p_1} + \varepsilon_2\|x_2\|^{p_2} + \cdots + \varepsilon_7\|x_7\|^{p_7} \\ &\le \mu\left[\varepsilon_1|||x_1|||^{p_1} + \varepsilon_2|||x_2|||^{p_2} + \cdots + \varepsilon_7|||x_7|||^{p_7}\right].\end{aligned}$$

According to the conditions of Corollary 2, we conclude

$$\|\phi(3^{-n}x_1, 3^{-n}x_2, \cdots, 3^{-n}x_7)\| \le \sum_{i=1}^{7} \varepsilon_i 3^{-n\beta p_i}\|x_i\|^{p_i}.$$

Since $p_i \in (\frac{1}{\beta}, +\infty)$ $(i = 1, 2, \cdots, 7)$,

$$\lim_{n\to\infty} \|3^n \phi(3^{-n}x_1, 3^{-n}x_2, \cdots, 3^{-n}x_7)\| \le \lim_{n\to\infty} \sum_{i=1}^{7} \varepsilon_i 3^{-n(\beta p_i - 1)} \|x\|^{p_i} = 0.$$

From the proofs of Theorem 3, we get

$$g(x) - 3^n g(3^{-n}x) = \sum_{k=1}^{n} 3^{k-1} \Psi\left(3^{-(k-1)}x\right).$$

As in the proofs of Theorem 3 and Corollary 1, we can achieve that

$$\sum_{k=1}^{\infty} 3^{k-1} \Psi\left(3^{-(k-1)}x\right)$$

exists for every $x \in G$. Combining the definition of $\Psi_1(x)$ and (60), we obtain

$$\begin{aligned}
&\|\Psi_1(3^{-(k-1)}x)\| \\
&= \|\phi\left(3^{-(k-1)}x, -3^{-(k-2)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x\right) \\
&\quad + \phi\left(-3^{-(k-1)}x, 3^{-(k-2)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x\right) \\
&\quad - \phi\left(-3^{-(k-1)}x, 3^{-(k-2)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x\right) \\
&\quad - \phi\left(3^{-(k-1)}x, -3^{-(k-2)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x\right)\| \\
&\le \|\phi\left(3^{-(k-1)}x, -3^{-(k-2)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x\right)\| \\
&\quad + \|\phi\left(-3^{-(k-1)}x, 3^{-(k-2)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x\right)\| \\
&\quad + \|\phi\left(-3^{-(k-1)}x, 3^{-(k-2)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x\right)\| \\
&\quad + \|\phi\left(3^{-(k-1)}x, -3^{-(k-2)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x\right)\| \\
&\le 4\left[\varepsilon_1 3^{-(k-1)\beta p_1} \|x\|^{p_1} + \varepsilon_2 3^{-(k-2)\beta p_2} \|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{-(k-1)\beta p_i} \|x\|^{p_i}\right].
\end{aligned}$$

As in the proof of the above, we may follow

$$\begin{aligned}
&\|\Psi_2(3^{-(k-1)}x)\| \\
&\le 4\left[\varepsilon_1 3^{-(k-1)\beta p_1} \|x\|^{p_1} + \varepsilon_2 3^{-(k-2)\beta p_2} \|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{-(k-1)\beta p_i} \|x\|^{p_i}\right],
\end{aligned}$$

$$\begin{aligned}
&\|\Psi_3(3^{-(k-1)}x)\| \\
&\leq 4\left[\varepsilon_1 3^{-(k-1)\beta p_1}\|x\|^{p_1} + \varepsilon_2 3^{-(k-2)\beta p_2}\|x\|^{p_2} + \sum_{i=3}^{7}\varepsilon_i 3^{-(k-1)\beta p_i}\|x\|^{p_i}\right], \\
&\|\Psi_4(3^{-(k-1)}x)\| \\
&\leq 4\left[\sum_{\substack{i=1\\i=6}}^{3}\varepsilon_i 3^{-(k-1)\beta p_i}\|x\|^{p_i} + 2^{\beta p_7}\times 3^{-(k-1)\beta p_7}\varepsilon_7\|x\|^{p_7}\right] \\
&+2\sum_{i=4}^{5}(2^{\beta p_i}+1)\varepsilon_i 3^{-(k-1)\beta p_i}\|x\|^{p_i}, \\
&\|\Psi_5(3^{-(k-1)}x)\| \leq 4\left[\varepsilon_1 3^{-(k-1)\beta p_1}\|x\|^{p_1} + \sum_{i=2}^{7}\varepsilon_i \frac{1}{2^{\beta p_i}} 3^{-(k-1)\beta p_i}\|x\|^{p_i}\right].
\end{aligned}$$

As a consequence,

$$\begin{aligned}
& \|T(x) - f(x) - f(\theta)\| \\
= \; & \|\sum_{k=1}^{\infty} 3^{k-1}\Psi\left(3^{-(k-1)}x\right)\| \leq \sum_{k=1}^{\infty} 3^{k-1}\|\Psi\left(3^{-(k-1)}x\right)\| \\
= \; & \sum_{k=1}^{\infty} 3^{k-1}\|\frac{1}{78}[13\Psi_1(3^{-(k-1)}x) + 15\Psi_2(3^{-(k=1)}x) - 3\Psi_3(3^{-(k-1)}x) \\
- \; & 6\Psi_4(3^{-(k-1)}x) - 12\Psi_5(3^{-(k-1)}x)]\| \\
\leq \; & \sum_{k=1}^{\infty} 3^{k-1}\left[\|\Psi_1(3^{-(k-1)}x)\| + \|\Psi_2(3^{-(k-1)}x)\| + \|\Psi_3(3^{-(k-1)}x)\|\right] \\
+ \; & \sum_{k=1}^{\infty} 3^{k-1}\left[\|\Psi_4(3^{-(k-1)}x)\| + \|\Psi_5(3^{-(k-1)}x)\|\right] \\
\leq \; & \sum_{k=1}^{\infty} 3^{-(k-1)(\beta p_1 - 1)} \times 4 \times 5\varepsilon_1\|x\|^{p_1} \\
+ \; & \sum_{k=1}^{\infty} 3^{-(k-1)(\beta p_2 - 1)} \times 4 \times \left[3^{\beta p_2 + 1} + 1 + \frac{1}{2^{\beta p_2}}\right]\varepsilon_2\|x\|^{p_2} \\
+ \; & \sum_{k=1}^{\infty} 3^{-(k-1)(\beta p_3 - 1)} \times 4 \times \left[4 + \frac{1}{2^{\beta p_3}}\right]\varepsilon_3\|x\|^{p_3}
\end{aligned}$$

$$\begin{aligned}
&+ \sum_{k=1}^{\infty} 3^{-(k-1)(\beta p_4-1)} \times 2 \times \left(7+2^{\beta p_4}+\frac{1}{2^{(\beta p_4-1)}}\right)\varepsilon_4\|x\|^{p_4}\\
&+ \sum_{k=1}^{\infty} 3^{-(k-1)(\beta p_5-1)} \times 2 \times \left(7+2^{\beta p_5}+\frac{1}{2^{(\beta p_5-1)}}\right)\varepsilon_5\|x\|^{p_5}\\
&+ \sum_{k=1}^{\infty} 3^{-(k-1)(\beta p_6-1)} \times 4 \times \left[4+\frac{1}{2^{\beta p_6}}\right]\varepsilon_6\|x\|^{p_6}\\
&+ \sum_{k=1}^{\infty} 3^{-(k-1)(\beta p_7-1)} \times 4 \times \left(3+2^{\beta p_7}+\frac{1}{2^{\beta p_7}}\right)\varepsilon_7\|x\|^{p_7}\\
&= \varepsilon_1 B_1\|x\|^{p_1}+\varepsilon_2 B_2\|x\|^{p_2}+\cdots+\varepsilon_7 B_7\|x\|^{p_7}.
\end{aligned}$$

Uniqueness: The proof of uniqueness is similar to the proof of Theorem 2, so we omit it.

Linearity: As in the proof of Corollary 1, it can be followed that $T(x)$ is linear under the assumption that $f(x)$ is continuous. □

Finally, we investigate the case of $\beta_2 < p_i < \frac{1}{\beta_1}$ ($0 < \beta_1, \beta_2 \le 1, i = 1, 2, \cdots, 7$).

Corollary 3. *Suppose that G is a β_1-homogeneous F^*-space and E a β_2-homogeneous F-space. Given $\varepsilon_i \ge 0$, $\beta_2 < p_i < \frac{1}{\beta_1}$ and $p_i \ne \frac{\beta_2}{\beta_1}$ ($i = 1, 2, \cdots, 7$), if ϕ satisfies*

$$\|\phi(x_1, x_2, \cdots, x_7)\| \le \varepsilon_1\|x_1\|^{p_1}+\varepsilon_2\|x_2\|^{p_2}+\cdots+\varepsilon_7\|x_7\|^{p_7} \quad (\forall x_i \in G, i=1,2,\cdots,7), \tag{61}$$

then there exists a unique additive mapping $T\colon G \to E$ such that

$$\|T(x)-f(x)+f(\theta)\| \le \varepsilon_1 B_1\|x\|^{p_1}+\varepsilon_2 B_2\|x\|^{p_2}+\cdots+\varepsilon_7 B_7\|x\|^{p_7} \quad (\forall x \in G).$$

When $p_i < \frac{\beta_2}{\beta_1}$ ($i = 1, 2, \cdots, 7$), we have

$$\begin{aligned}
B_1 &\overset{\text{def}}{=} \frac{4\times 3^{p_1\beta_1}}{78^{\beta_2}(3^{\beta_2}-3^{p_1\beta_1})}\left(13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+6^{\beta_2}+12^{\beta_2}\right),\\
B_2 &\overset{\text{def}}{=} \frac{4\times 3^{p_2\beta_1}}{78^{\beta_2}(3^{\beta_2}-3^{p_2\beta_1})}\left[3^{p_2}\times\left(13^{\beta_2}+15^{\beta_2}+3^{\beta_2}\right)+6^{\beta_2}+\frac{12^{\beta_2}}{2^{p_2\beta_1}}\right],\\
B_3 &\overset{\text{def}}{=} \frac{4\times 3^{p_3\beta_1}}{78^{\beta_2}(3^{\beta_2}-3^{p_3\beta_1})}\left(13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+6^{\beta_2}+\frac{12^{\beta_2}}{2^{p_3\beta_1}}\right),
\end{aligned}$$

$$B_4 \overset{\text{def}}{=} \frac{2\times 3^{p_4\beta_1}}{78^{\beta_2}(3^{\beta_2}-3^{p_4\beta_1})}\left[2\left(13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+\frac{12^{\beta_2}}{2^{p_4\beta_1}}\right)+\left(2^{p_4\beta_1}+1\right)6^{\beta_2}\right],$$

$$B_5 \overset{\text{def}}{=} \frac{2\times 3^{p_5\beta_1}}{78^{\beta_2}(3^{\beta_2}-3^{p_5\beta_1})}\left[2\left(13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+\frac{12^{\beta_2}}{2^{p_5\beta_1}}\right)+\left(2^{p_5\beta_1}+1\right)6^{\beta_2}\right],$$

$$B_6 \overset{\text{def}}{=} \frac{4\times 3^{p_6\beta_1}}{78^{\beta_2}(3^{\beta_2}-3^{p_6\beta_1})}\left(13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+6^{\beta_2}+\frac{12^{\beta_2}}{2^{p_6\beta_1}}\right),$$

$$B_7 \overset{\text{def}}{=} \frac{4\times 3^{p_7\beta_1}}{78^{\beta_2}(3^{\beta_2}-3^{p_7\beta_1})}\left(13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+6^{\beta_2}\times 2^{p_7^{\beta_1}}+\frac{12^{\beta_2}}{2^{p_7\beta_1}}\right).$$

When $p_i > \frac{\beta_2}{\beta_1}$ $(\ i=1,2,\cdots,7\ )$, *we have*

$$B_1 \overset{\text{def}}{=} \frac{20\times 3^{\beta_1 p_1}}{3^{\beta_1 p_1}-3^{\beta_2}}, \qquad B_2 \overset{\text{def}}{=} \frac{4\times 3^{\beta_1 p_2}}{3^{\beta_1 p_2}-3^{\beta_2}}\left(3^{\beta_1 p_2+1}+1+\frac{1}{2^{\beta_1 p_2}}\right),$$

$$B_3 \overset{\text{def}}{=} \frac{4\times 3^{\beta_1 p_3}}{3^{\beta_1 p_3}-3^{\beta_2}}\left(4+\frac{1}{2^{\beta_1 p_3}}\right), \qquad B_4 \overset{\text{def}}{=} \frac{2\times 3^{\beta_1 p_4}}{3^{\beta_1 p_4}-3^{\beta_2}}\left(7+2^{\beta_1 p_4}+\frac{2}{2^{\beta_1 p_4}}\right),$$

$$B_5 \overset{\text{def}}{=} \frac{2\times 3^{\beta_1 p_5}}{3^{\beta_1 p_5}-3^{\beta_2}}\left(7+2^{\beta_1 p_5}+\frac{2}{2^{\beta_1 p_5}}\right), \qquad B_6 \overset{\text{def}}{=} \frac{4\times 3^{\beta_1 p_6}}{3^{\beta_1 p_6}-3^{\beta_2}}\left(4+\frac{1}{2^{\beta_1 p_6}}\right),$$

$$B_7 \overset{\text{def}}{=} \frac{4\times 3^{\beta_1 p_7}}{3^{\beta_1 p_7}-3^{\beta_2}}\left(3+2^{\beta_1 p_7}+\frac{1}{2^{\beta_1 p_7}}\right).$$

Especially, $T(x)$ *is linear under the condition that* $f(x)$ *is continuous.*

Proof. Let $g(x)=f(x)-f(\theta)$ for any $x\in G$.

At first, we prove the case of $p_i < \frac{\beta_2}{\beta_1}$ $(\ i=1,2,\cdots,7\ )$.

Firstly, we prove the existence of T. According to (61), we get that

$$\begin{aligned}&\|\phi\left(3^n x_1, 3^n x_2, \cdots, 3^n x_7\right)\| \\ &\le \varepsilon_1 3^{np_1\beta_1}\|x_1\|^{p_1}+\varepsilon_2 3^{np_2\beta_1}\|x_2\|^{p_2}+\cdots+\varepsilon_7 3^{np_7\beta_1}\|x_7\|^{p_7}.\end{aligned}$$

Since $p_i < \frac{\beta_2}{\beta_1}$ $(\ i=1,2,\cdots,7\ )$, we have

$$\lim_{n\to\infty}\|\frac{\phi\left(3^n x_1, 3^n x_2, \cdots, 3^n x_7\right)}{3^n}\| \le \lim_{n\to\infty}\sum_{i=1}^{7}\frac{1}{3^{n(\beta_2-p_i\beta_1)}}\varepsilon_i\|x_i\|^{p_i}=0.$$

Furthermore, by Corollary 1 we have

$$\frac{1}{3^n}g(3^n x)-g(x)=\sum_{k=1}^{n}\frac{\Psi(3^k x)}{3^k}$$

holds for any $n \in \mathbb{N}$, where Ψ is as above. It is clear that $\sum_{k=1}^{\infty} \frac{\Psi(3^k x)}{3^k}$ exits for every $x \in G$. Indeed, from Corollary 1, we can get

$$\frac{g(3^m x)}{3^m} - \frac{g(3^n x)}{3^n} = \sum_{k=n+1}^{m} \frac{\Psi(3^k x)}{3^k}.$$

Combining (61) and β_1-homogeneity of norm in G, we have

$$\begin{aligned}
&\|\Psi_1(3^k x)\| \\
&\leq \|\phi\left(3^k x, -3^{k+1}x, 3^k x, -3^k x, 3^k x, -3^k x, 3^k x\right)\| \\
&\quad + \|\phi\left(-3^k x, 3^{k+1}x, -3^k x, 3^k x, -3^k x, -3^k x, 3^k x\right)\| \\
&\quad + \|\phi\left(-3^k x, 3^{k+1}x, -3^k x, 3^k x, -3^k x, 3^k x, -3^k x\right)\| \\
&\quad + \|\phi\left(3^k x, -3^{k+1}x, 3^k x, -3^k x, 3^k x, 3^k x, -3^k x\right)\| \\
&\leq 4\left[\varepsilon_1 3^{kp_1\beta_1}\|x\|^{p_1} + \varepsilon_2 3^{(k+1)p_2\beta_1}\|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{kp_i\beta_1}\|x\|^{p_i}\right].
\end{aligned}$$

As in the above proof, we can get

$$\begin{aligned}
\|\Psi_2(3^k x)\| &\leq 4\left[\varepsilon_1 3^{kp_1\beta_1}\|x\|^{p_1} + \varepsilon_2 3^{(k+1)p_2\beta_1}\|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{kp_i\beta_1}\|x\|^{p_i}\right], \\
\|\Psi_3(3^k x)\| &\leq 4\left[\varepsilon_1 3^{kp_1\beta_1}\|x\|^{p_1} + \varepsilon_2 3^{(k+1)p_2\beta_1}\|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{kp_i\beta_1}\|x\|^{p_i}\right], \\
\|\Psi_4(3^k x)\| &\leq 4\left[\sum_{i=1}^{3} \varepsilon_i 3^{kp_i\beta_1}\|x\|^{p_i} + \varepsilon_6 3^{kp_6\beta_1}\|x\|^{p_6} + \varepsilon_7 2^{p_7\beta_1} \times 3^{kp_7\beta_1}\|x\|^{p_7}\right] \\
&\quad + 2\sum_{i=4}^{5} \varepsilon_i \left(2^{p_i\beta_1} + 1\right) \times 3^{kp_i\beta_1}\|x\|^{p_i}, \\
\|\Psi_5(3^k x)\| &\leq 4\left[\varepsilon_1 3^{kp_1\beta_1}\|x\|^{p_1} + \sum_{i=2}^{7} \varepsilon_i 3^{kp_i\beta_1} \times \frac{1}{2^{p_i\beta_1}}\|x\|^{p_i}\right].
\end{aligned}$$

Furthermore, for any $m > n$, $m, n \in \mathbb{N}$, we have

$$\left\|\frac{g(3^m x)}{3^m} - \frac{g(3^n x)}{3^n}\right\| = \left\|\sum_{k=n+1}^{m} \frac{\Psi(3^k x)}{3^k}\right\| \leq \sum_{k=n+1}^{m} \left\|\frac{\Psi(3^k x)}{3^k}\right\|$$

$$
\begin{aligned}
&= \sum_{k=n+1}^{m} \left\| \frac{1}{3^k 78} \left[13\Psi_1(3^k x) + 15\Psi_2(3^k x) - 3\Psi_3(3^k x) - 6\Psi_4(x) - 12\Psi_5(3^k x) \right] \right\| \\
&\leq \sum_{k=n+1}^{m} \frac{1}{3^{k\beta_2} 78^{\beta_2}} \left[13^{\beta_2} \|\Psi_1(3^k x)\| + 15^{\beta_2} \|\Psi_2(3^k x)\| + 3^{\beta_2} \|\Psi_3(3^k x)\| \right] \\
&+ \sum_{k=n+1}^{m} \frac{1}{3^{k\beta_2} 78^{\beta_2}} \left[6^{\beta_2} \|\Psi_4(3^k x)\| + 12^{\beta_2} \|\Psi_5(3^k x)\| \right] \\
&\leq \sum_{k=n+1}^{m} \frac{4 \times 13^{\beta_2}}{3^{k\beta_2} \times 78^{\beta_2}} \left[\varepsilon_1 3^{kp_1\beta_1} \|x\|^{p_1} + \varepsilon_2 3^{(k+1)p_2\beta_1} \|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{kp_i\beta_1} \|x\|^{p_i} \right] \\
&+ \sum_{k=n+1}^{m} \frac{4 \times 15^{\beta_2}}{3^{k\beta_2} \times 78^{\beta_2}} \left[\varepsilon_1 3^{kp_1\beta_1} \|x\|^{p_1} + \varepsilon_2 3^{(k+1)p_2\beta_1} \|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{kp_i\beta_1} \|x\|^{p_i} \right] \\
&+ \sum_{k=n+1}^{m} \frac{4 \times 3^{\beta_2}}{3^{k\beta_2} \times 78^{\beta_2}} \left[\varepsilon_1 3^{kp_1\beta_1} \|x\|^{p_1} + \varepsilon_2 3^{(k+1)p_2\beta_1} \|x\|^{p_2} + \sum_{i=3}^{7} \varepsilon_i 3^{kp_i\beta_1} \|x\|^{p_i} \right] \\
&+ \sum_{k=n+1}^{m} \frac{4 \times 6^{\beta_2}}{3^{k\beta_2} \times 78^{\beta_2}} \left[\sum_{i=1}^{3} \varepsilon_i 3^{kp_i\beta_1} \|x\|^{p_i} + \varepsilon_6 3^{kp_6\beta_1} \|x\|^{p_6} + \varepsilon_7 2^{p_7\beta_1} \times 3^{kp_7\beta_1} \|x\|^{p_7} \right] \\
&+ \sum_{k=n+1}^{m} \frac{2 \times 6^{\beta_2}}{3^{k\beta_2} \times 78^{\beta_2}} \left[\sum_{i=4}^{5} \varepsilon_i \left(2^{p_i\beta_1} + 1 \right) \times 3^{kp_i\beta_1} \|x\|^{p_i} \right] \\
&+ \sum_{k=n+1}^{m} \frac{4 \times 12^{\beta_2}}{3^{k\beta_2} \times 78^{\beta_2}} \left[\varepsilon_1 3^{kp_1\beta_1} \|x\|^{p_1} + \sum_{i=2}^{7} \varepsilon_i 3^{kp_i\beta_1} \times \frac{1}{2^{p_i\beta_1}} \|x\|^{p_i} \right] \\
&= \sum_{k=n+1}^{m} \frac{4}{3^{k(\beta_2 - p_1\beta_1)} \times 78^{\beta_2}} \left[13^{\beta_2} + 15^{\beta_2} + 3^{\beta_2} + 6^{\beta_2} + 12^{\beta_2} \right] \times \varepsilon_1 \times \|x\|^{p_1} \\
&+ \sum_{k=n+1}^{m} \frac{4 \times \varepsilon_2 \times \|x\|^{p_2}}{3^{k(\beta_2 - p_2\beta_1)} \times 78^{\beta_2}} \left[\left(13^{\beta_2} + 15^{\beta_2} + 3^{\beta_2} \right) \times 3^{p_2\beta_1} + 6^{\beta_2} + \frac{12^{\beta_2}}{2^{p_2\beta_1}} \right] \\
&+ \sum_{k=n+1}^{m} \frac{4}{3^{k(\beta_2 - p_3\beta_1)} \times 78^{\beta_2}} \left[13^{\beta_2} + 15^{\beta_2} + 3^{\beta_2} + 6^{\beta_2} + \frac{12^{\beta_2}}{2^{p_3\beta_1}} \right] \times \varepsilon_3 \times \|x\|^{p_3} \\
&+ \sum_{k=n+1}^{m} \frac{\varepsilon_4 \times \|x\|^{p_4}}{3^{k(\beta_2 - p_4\beta_1)} \times 78^{\beta_2}} \left[4 \left(13^{\beta_2} + 15^{\beta_2} + 3^{\beta_2} + \frac{12^{\beta_2}}{2^{p_4\beta_1}} \right) + 2(2^{p_4\beta_1} + 1) 6^{\beta_2} \right] \\
&+ \sum_{k=n+1}^{m} \frac{\varepsilon_5 \times \|x\|^{p_5}}{3^{k(\beta_2 - p_5\beta_1)} \times 78^{\beta_2}} \left[4 \left(13^{\beta_2} + 15^{\beta_2} + 3^{\beta_2} + \frac{12^{\beta_2}}{2^{p_5\beta_1}} \right) + 2(2^{p_5\beta_1} + 1) 6^{\beta_2} \right]
\end{aligned}
$$

$$+\sum_{k=n+1}^{m}\frac{4\times\varepsilon_6\times\|x\|^{p_6}}{3^{k(\beta_2-p_6\beta_1)}\times 78^{\beta_2}}\left[13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+6^{\beta_2}+\frac{12^{\beta_2}}{2^{p_6\beta_1}}\right]$$
$$+\sum_{k=n+1}^{m}\frac{4\times\varepsilon_7\times\|x\|^{p_7}}{3^{k(\beta_2-p_7\beta_1)}\times 78^{\beta_2}}\left[13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+6^{\beta_2}\times 2^{p_7}+\frac{12^{\beta_2}}{2^{p_7\beta_1}}\right].$$

Hence, $\left\{\frac{g(3^n x)}{3^n}\right\}$ is Cauchy in E because of $p_i<\frac{\beta_2}{\beta_1}$ ($i=1,2,\cdots,7$). It converges to an element of E since E is complete. From (55), we infer

$$\sum_{k=1}^{\infty}\frac{\Psi(3^k x)}{3^k}$$

exists for every $x\in G$. Thus by Theorem 2,

$$T(x)=\lim_{n\to\infty}\frac{g(3^n x)}{3^n}=\lim_{n\to\infty}\frac{g(3^n x)+f(\theta)}{3^n}=\lim_{n\to\infty}\frac{f(3^n x)}{3^n}$$

and it is additive. In addition, $T(x)-f(x)+f(\theta)=\sum\limits_{k=1}^{\infty}\frac{\Psi(3^k x)}{3^k}$ follows from (55). Furthermore,

$$\|T(x)-f(x)+f(\theta)\|=\|\sum_{k=1}^{\infty}\frac{\Psi(3^k x)}{3^k}\|$$
$$\leq\sum_{k=1}^{\infty}\frac{4\times\varepsilon_1\times\|x\|^{p_1}}{3^{k(\beta_2-p_1\beta_1)}\times 78^{\beta_2}}\left[13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+6^{\beta_2}+12^{\beta_2}\right]$$
$$+\sum_{k=1}^{\infty}\frac{4\times\varepsilon_2\times\|x\|^{p_2}}{3^{k(\beta_2-p_2\beta_1)}\times 78^{\beta_2}}\left[\left(13^{\beta_2}+15^{\beta_2}+3^{\beta_2}\right)\times 3^{p_2\beta_1}+6^{\beta_2}+\frac{12^{\beta_2}}{2^{p_2\beta_1}}\right]$$
$$+\sum_{k=1}^{\infty}\frac{4\times\varepsilon_3\times\|x\|^{p_3}}{3^{k(\beta_2-p_3\beta_1)}\times 78^{\beta_2}}\left[13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+6^{\beta_2}+\frac{12^{\beta_2}}{2^{p_3\beta_1}}\right]$$
$$+\sum_{k=1}^{\infty}\frac{\varepsilon_4\times\|x\|^{p_4}}{3^{k(\beta_2-p_4\beta_1)}\times 78^{\beta_2}}\left[4\left(13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+\frac{12^{\beta_2}}{2^{p_4\beta_1}}\right)+2(2^{p_4\beta_1}+1)\times 6^{\beta_2}\right]$$
$$+\sum_{k=1}^{\infty}\frac{\varepsilon_5\times\|x\|^{p_5}}{3^{k(\beta_2-p_5\beta_1)}\times 78^{\beta_2}}\left[4\left(13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+\frac{12^{\beta_2}}{2^{p_5\beta_1}}\right)+2(2^{p_5\beta_1}+1)\times 6^{\beta_2}\right]$$
$$+\sum_{k=1}^{\infty}\frac{4\times\varepsilon_6\times\|x\|^{p_6}}{3^{k(\beta_2-p_6\beta_1)}\times 78^{\beta_2}}\left[13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+6^{\beta_2}+\frac{12^{\beta_2}}{2^{p_6\beta_1}}\right]$$

$$+\sum_{k=1}^{\infty}\frac{4\times\varepsilon_7\times\|x\|^{p_7}}{3^{k(\beta_2-p_7\beta_1)}\times 78^{\beta_2}}\left[13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+6^{\beta_2}\times 2^{p_7\beta_1}+\frac{12^{\beta_2}}{2^{p_7\beta_1}}\right]$$
$$=\varepsilon_1 B_1\|x\|^{p_1}+\varepsilon_2 B_2\|x\|^{p_2}+\cdots+\varepsilon_7 B_7\|x\|^{p_7}$$

for all $x \in G$.

Secondly, we prove the uniqueness of T. If $U\colon G \to E$ is another additive mapping satisfying

$$\|U(x)-f(x)+f(\theta)\| \le \varepsilon_1 B_1\|x\|^{p_1}+\varepsilon_2 B_2\|x\|^{p_2}+\cdots+\varepsilon_7 B_7\|x\|^{p_7}$$

for all $x \in G$, it follows from the last two inequalities that

$$\begin{aligned}
&\|U(x)-T(x)\| = \frac{1}{n^{\beta_2}}\|U(nx)-T(nx)\| \\
= \;& \frac{1}{n^{\beta_2}}\|U(nx)-f(nx)+f(\theta)-T(nx)+f(nx)-f(\theta)\| \\
\le \;& \frac{1}{n^{\beta_2}}\left[\|U(nx)-f(nx)+f(\theta)\|+\|T(nx)-f(nx)+f(\theta)\|\right] \\
\le \;& \frac{2}{n^{\beta_2}}(\varepsilon_1 B_1\|nx\|^{p_1}+\varepsilon_2 B_2\|nx\|^{p_2}+\cdots+\varepsilon_7 B_7\|nx\|^{p_7}) \\
= \;& 2\left[\frac{\varepsilon_1 B_1}{n^{\beta_2-p_1\beta_1}}\|x\|^{p_1}+\frac{\varepsilon_2 B_2}{n^{\beta_2-p_2\beta_1}}\|x\|^{p_2}+\cdots+\frac{\varepsilon_7 B_7}{n^{\beta_2-p_7\beta_1}}\|x\|^{p_7}\right].
\end{aligned}$$

Taking the limit as $n \to \infty$, we get $\|U(x)-T(x)\| \to 0$ since $p_i < \frac{\beta_2}{\beta_1}$ ($i = 1, 2, \cdots, 7$). As a result, $U(x) = T(x)$ for all $x \in G$.

Linearity: If $f(x)$ is continuous in G, then $T_n(x) = \frac{f(3^n x)}{3^n}$ is continuous in G for any $n \in \mathbb{N}$. It can be seen that $\{T_n(x)\}$ uniformly converges to $T(x)$ in each closed ball from the above proof. So $T(x)$ is continuous. It concludes $T(x)$ is linear. Then, we give the proofs about the case of $p_i > \frac{\beta_2}{\beta_1}$ ($i = 1, 2, \cdots, 7$).

Existence: Similar to Corollary 2, we may assume that F-norm $\|\cdot\|$ of E is non-decreasing. According to the conditions of Corollary 3, we conclude

$$\|\phi(3^{-n}x_1, 3^{-n}x_2, \cdots, 3^{-n}x_7)\| \le \sum_{i=1}^{7}\varepsilon_i 3^{-n\beta_1 p_i}\|x_i\|^{p_i}.$$

Since $p_i > \frac{\beta_2}{\beta_1}$ ($i = 1, 2, \cdots, 7$),

$$\lim_{n\to\infty}\|3^n\phi(3^{-n}x_1, 3^{-n}x_2, \cdots, 3^{-n}x_7)\| \le \lim_{n\to\infty}\sum_{i=1}^{7}\varepsilon_i 3^{-n(\beta_1 p_i-\beta_2)}\|x\|^{p_i} = 0.$$

From the proofs of Theorem 3, we get

$$g(x) - 3^n g(3^{-n}x) = \sum_{k=1}^{n} 3^{k-1}\Psi\left(3^{-(k-1)}x\right).$$

As in the proofs of $p_i < \frac{\beta_2}{\beta_1}$ ($i = 1, 2, \cdots, 7$), we can achieve that

$$\sum_{k=1}^{\infty} 3^{k-1}\Psi\left(3^{-(k-1)}x\right)$$

exists for every $x \in G$. Combining the definition of $\Psi_1(x)$ and (60), we obtain

$$\begin{aligned}
&\|\Psi_1(3^{-(k-1)}x)\| \\
&= \|\phi\Big(3^{-(k-1)}x, -3^{-(k-2)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x\Big) \\
&+\phi\Big(-3^{-(k-1)}x, 3^{-(k-2)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x\Big) \\
&-\phi\Big(-3^{-(k-1)}x, 3^{-(k-2)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x\Big) \\
&-\phi\Big(3^{-(k-1)}x, -3^{-(k-2)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x\Big)\| \\
&\le \|\phi\Big(3^{-(k-1)}x, -3^{-(k-2)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x\Big)\| \\
&+\|\phi\Big(-3^{-(k-1)}x, 3^{-(k-2)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x\Big)\| \\
&+\|\phi\Big(-3^{-(k-1)}x, 3^{-(k-2)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x\Big)\| \\
&+\|\phi\Big(3^{-(k-1)}x, -3^{-(k-2)}x, 3^{-(k-1)}x, -3^{-(k-1)}x, 3^{-(k-1)}x, 3^{-(k-1)}x, -3^{-(k-1)}x\Big)\| \\
&\le 4\left[\varepsilon_1 3^{-(k-1)\beta_1 p_1}\|x\|^{p_1} + \varepsilon_2 3^{-(k-2)\beta_1 p_2}\|x\|^{p_2} + \sum_{i=3}^{7}\varepsilon_i 3^{-(k-1)\beta_1 p_i}\|x\|^{p_i}\right],
\end{aligned}$$

As in the proofs of the above, we may follow

$$\begin{aligned}
&\|\Psi_2(3^{-(k-1)}x)\| \\
&\le 4\left[\varepsilon_1 3^{-(k-1)\beta_1 p_1}\|x\|^{p_1} + \varepsilon_2 3^{-(k-2)\beta_1 p_2}\|x\|^{p_2} + \sum_{i=3}^{7}\varepsilon_i 3^{-(k-1)\beta_1 p_i}\|x\|^{p_i}\right], \\
&\|\Psi_3(3^{-(k-1)}x)\| \\
&\le 4\left[\varepsilon_1 3^{-(k-1)\beta_1 p_1}\|x\|^{p_1} + \varepsilon_2 3^{-(k-2)\beta_1 p_2}\|x\|^{p_2} + \sum_{i=3}^{7}\varepsilon_i 3^{-(k-1)\beta_1 p_i}\|x\|^{p_i}\right],
\end{aligned}$$

$$\|\Psi_4(3^{-(k-1)}x)\| \leq 4\left[\sum_{\substack{i=1\\i=6}}^{3}\varepsilon_i 3^{-(k-1)\beta_1 p_i}\|x\|^{p_i} + 2^{\beta_1 p_7}3^{-(k-1)\beta_1 p_7}\varepsilon_7\|x\|^{p_7}\right]$$
$$+2\sum_{i=4}^{5}(2^{\beta_1 p_i}+1)\varepsilon_i 3^{-(k-1)\beta_1 p_i}\|x\|^{p_i},$$
$$\|\Psi_5(3^{-(k-1)}x)\| \leq 4\left[\varepsilon_1 3^{-(k-1)\beta_1 p_1}\|x\|^{p_1} + \sum_{i=2}^{7}\varepsilon_i\frac{1}{2^{\beta_1 p_i}}3^{-(k-1)\beta_1 p_i}\|x\|^{p_i}\right].$$

As a consequence,

$$\|T(x)-f(x)-f(\theta)\| = \|\sum_{k=1}^{\infty}3^{k-1}\Psi\left(3^{-(k-1)}x\right)\|$$
$$\leq \sum_{k=1}^{\infty}3^{(k-1)\beta_2}\|\Psi\left(3^{-(k-1)}x\right)\|$$
$$= \sum_{k=1}^{\infty}3^{(k-1)\beta_2}\|\frac{1}{78}[13\Psi_1(3^{-(k-1)}x)+15\Psi_2(3^{-(k-1)}x)-3\Psi_3(3^{-(k-1)}x)$$
$$- 6\Psi_4(3^{-(k-1)}x)-12\Psi_5(3^{-(k-1)}x)]\|$$
$$\leq \sum_{k=1}^{\infty}3^{(k-1)\beta_2}\left[\|\Psi_1(3^{-(k-1)}x)\|+\|\Psi_2(3^{-(k-1)}x)\|+\|\Psi_3(3^{-(k-1)}x)\|\right]$$
$$+ \sum_{k=1}^{\infty}3^{(k-1)\beta_2}\left[\|\Psi_4(3^{-(k-1)}x)\|+\|\Psi_5(3^{-(k-1)}x)\|\right]$$
$$\leq \sum_{k=1}^{\infty}3^{-(k-1)(\beta_1 p_1-\beta_2)}\times 4\times 5\varepsilon_1\|x\|^{p_1}$$
$$+ \sum_{k=1}^{\infty}3^{-(k-1)(\beta_1 p_2-\beta_2)}\times 4\times\left[3^{\beta_1 p_2+1}+1+\frac{1}{2^{\beta_1 p_2}}\right]\varepsilon_2\|x\|^{p_2}$$
$$+ \sum_{k=1}^{\infty}3^{-(k-1)(\beta_1 p_3-\beta_2)}\times 4\times\left[4+\frac{1}{2^{\beta_1 p_3}}\right]\varepsilon_3\|x\|^{p_3}$$
$$+ \sum_{k=1}^{\infty}3^{-(k-1)(\beta_1 p_4-\beta_2)}\times 2\times\left(7+2^{\beta_1 p_4}+\frac{2}{2^{\beta_1 p_4}}\right)\varepsilon_4\|x\|^{p_4}$$
$$+ \sum_{k=1}^{\infty}3^{-(k-1)(\beta_1 p_5-\beta_2)}\times 2\times\left(7+2^{\beta_1 p_5}+\frac{2}{2^{\beta_1 p_5}}\right)\varepsilon_5\|x\|^{p_5}$$

$$\begin{aligned}
&+ \sum_{k=1}^{\infty} 3^{-(k-1)(\beta_1 p_6-\beta_2)} \times 4 \times \left[4+\frac{1}{2^{\beta_1 p_6}}\right] \varepsilon_6 \|x\|^{p_6} \\
&+ \sum_{k=1}^{\infty} 3^{-(k-1)(\beta_1 p_7-\beta_2)} \times 4 \times \left(3+2^{\beta_1 p_7}+\frac{1}{2^{\beta_1 p_7}}\right) \varepsilon_7 \|x\|^{p_7} \\
&= \varepsilon_1 B_1 \|x\|^{p_1}+\varepsilon_2 B_2 \|x\|^{p_2}+\cdots+\varepsilon_7 B_7 \|x\|^{p_7}.
\end{aligned}$$

Uniqueness: If $H\colon G \to E$ is an another additive mapping satisfying

$$\|H(x)-f(x)+f(\theta)\| \le \varepsilon_1 B_1 \|x\|^{p_1}+\varepsilon_2 B_2 \|x\|^{p_2}+\cdots+\varepsilon_7 B_7 \|x\|^{p_7}$$

for all $x \in G$, it follows from the last two inequalities that

$$\begin{aligned}
&\|H(x)-T(x)\| = n^{\beta_2} \|U(\frac{x}{n})-T(\frac{x}{n})\| \\
&= \frac{1}{n^{\beta_2}} \|U(\frac{x}{n})-f(\frac{x}{n})+f(\theta)-T(\frac{x}{n})+f(\frac{x}{n})-f(\theta)\| \\
&\le \frac{1}{n^{\beta_2}} \left[\|U(\frac{x}{n})-f(\frac{x}{n})+f(\theta)\|+\|T(\frac{x}{n})-f(\frac{x}{n})+f(\theta)\|\right] \\
&\le \frac{2}{n^{\beta_2}} (\varepsilon_1 B_1 \|\frac{x}{n}\|^{p_1}+\varepsilon_2 B_2 \|\frac{x}{n}\|^{p_2}+\cdots+\varepsilon_7 B_7 \|\frac{x}{n}\|^{p_7}) \\
&= 2\varepsilon_1 B_1 n^{(\beta_2-p_1\beta_1)} \|x\|^{p_1}+2\varepsilon_2 B_2 n^{(\beta_2-p_2\beta_1)} \|x\|^{p_2} \\
&\quad +\cdots+2\varepsilon_7 B_7 n^{(\beta_2-p_7\beta_1)} \|x\|^{p_7}.
\end{aligned}$$

Taking the limit as $n \to \infty$, we get $\|H(x)-T(x)\| \to 0$ since $p_i > \frac{\beta_2}{\beta_1}$ ($i = 1, 2, \cdots, 7$). As a result, $H(x) = T(x)$ for all $x \in G$.

Linearity: If $f(x)$ is continuous in G, then $T_n(x) = 3^n f(3^{-n}x)$ is continuous in G for any $n \in \mathbb{N}$. It is easy to see that $\{T_n(x)\}$ uniformly converges to $T(x)$ in each closed ball from the above proof. As a result, $T(x)$ is continuous. It concludes $T(x)$ is linear. □

Remark 1. Let G, E, ε_i, δ, p_i be as in Corollary 3 and $p_i < \frac{\beta_2}{\beta_1}$ ($i = 1, 2, \cdots, 7$). If ϕ satisfies

$$\|\phi(x_1, x_2, \cdots, x_7)\| \le \delta+\varepsilon_1 \|x_1\|^{p_1}+\varepsilon_2 \|x_2\|^{p_2}+\cdots+\varepsilon_7 \|x_7\|^{p_7}$$

($\forall x_i \in G, i = 1, 2, \cdots, 7$) then there exists a unique additive mapping $T\colon G \to E$ such that

$$\|T(x)-f(x)+f(\theta)\| \le A\delta+\varepsilon_1 B_1 \|x\|^{p_1}+\varepsilon_2 B_2 \|x\|^{p_2}+\cdots+\varepsilon_7 B_7 \|x\|^{p_7} \quad (\forall x \in G).$$

$A \overset{\mathrm{def}}{=} \frac{4}{78^{\beta_2}(3^{\beta_2}-1)} \left(13^{\beta_2}+15^{\beta_2}+3^{\beta_2}+6^{\beta_2}+12^{\beta_2}\right)$ and B_i ($i = 1, 2, \cdots, 7$) are the same as the corresponding part of Corollary 3.

Remark 2. Let $G,\ E$ be as in Corollary 3 and f as in Theorem 4. Then

$$\|\phi(x_1, x_2, \cdots, x_7)\|_2 \leq 58^{\beta_2}\left(\|x_1\|_1^{\frac{\beta_2}{\beta_1}} + \|x_2\|_1^{\frac{\beta_2}{\beta_1}} + \cdots + \|x_7\|_1^{\frac{\beta_2}{\beta_1}}\right)$$

for any $x_i \in G, (\ i = 1, 2, \cdots, 7\ )$, but

$$\sup\left\{\frac{\|f(x) - T(x)\|_2}{\|x\|_1^{\frac{\beta_2}{\beta_1}}} :\ x \in \mathbb{R}\backslash\{0\}\right\} = \infty$$

for each additive mapping $T\colon G \to E$.

Rassias and Šemrl [12] have constructed a function

$$f\colon \mathbb{R} \to \mathbb{R}(f(x) \stackrel{\text{def}}{=} x\log_2(1 + |x|))$$

to show that the functional inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \gamma(\|x\| + \|y\|)$$

does not have Hyers-Ulam stability. By virtue of their method, we give a counterexample.

Theorem 4 *The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) \stackrel{\text{def}}{=} x\log_2(1+|x|)$ satisfies the inequality*

$$|\phi(x_1, x_2, \cdots, x_7)| \leq 58(|x_1| + |x_2| + \cdots + |x_7|) \quad (\ \forall x_i \in \mathbb{R},\ i = 1, 2, \cdots, 7)$$

but

$$\sup\left\{\left|\frac{f(x) - T(x)}{x}\right| :\ x \in \mathbb{R}\backslash\{0\}\right\} = \infty$$

for each additive mapping $T\colon \mathbb{R} \to \mathbb{R}$.

we can apply the same argument as in [5].

Remark 3. Let f be as of Theorem 4.

(i) If $G = (\mathbb{R}, \|\cdot\|_1)$ with the Euclidean metric $\|\cdot\|_1 = |\cdot|$ and $E = (\mathbb{R}, \|\cdot\|_2)$ with the β-homogeneous norm $\|\cdot\|_2 = |\cdot|^\beta$, then

$$\|\phi(x_1, x_2, \cdots, x_7)\|_2 \leq 58^\beta\left(\|x_1\|_1^\beta + \|x_2\|_1^\beta + \cdots + \|x_7\|_1^\beta\right)$$

$(\forall x_i \in G, \imath = 1, 2, \cdots, 7)$, but

$$\sup\left\{\frac{\|f(x) - T(x)\|_2}{\|x\|_1^\beta} :\ x \in \mathbb{R}\backslash\{0\}\right\} = \infty$$

for each additive mapping $T\colon G \to E$.

(ii) If $G = (\mathbb{R}, \|\cdot\|_1)$ with the β-homogeneous norm $\|\cdot\|_1 = |\cdot|^\beta$ and $E = (\mathbb{R}, \|\cdot\|_2)$ with the Euclidean metric $\|\cdot\|_2 = |\cdot|$, then

$$\|\phi(x_1, x_2, \cdots, x_7)\|_2 \leq 58\left(\|x_1\|_1^{\frac{1}{\beta}} + \|x_2\|_1^{\frac{1}{\beta}} + \cdots + \|x_7\|_1^{\frac{1}{\beta}}\right)$$

$(\forall x_i \in G,\ i = 1, 2, \cdots, 7)$, but

$$\sup\left\{\frac{\|f(x) - T(x)\|_2}{\|x\|_1^{\frac{1}{\beta}}} :\ x \in \mathbb{R}\backslash\{0\}\right\} = \infty$$

for each additive mapping $T\colon G \to E$.

(iii) If $G = E = (\mathbb{R}, \|\cdot\|)$ with the β-homogeneous norm $\|\cdot\| = |\cdot|^\beta$, then

$$\|\phi(x_1, x_2, \cdots, x_7)\| \leq 58^\beta(\|x_1\| + \|x_2\| + \cdots + \|x_7\|) \quad (\forall x_i \in G, i = 1, 2, \cdots, 7),$$

but

$$\sup\left\{\left\|\frac{f(x) - T(x)}{x}\right\| :\ x \in \mathbb{R}\backslash\{0\}\right\} = \infty$$

for each additive mapping $T\colon G \to E$.

References

[1] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184**(1994), 431–436.

[2] R. B. Holmes, *Geometric functional analysis and its applications*, Springer-Verlag, New York, 1975.

[3] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27**(1941), 222–224.

[4] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhauser, Boston, Basel, Berlin, 1998.

[5] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Jensen's equation and its application*, Proc. Amer. Math. Soc., **126**(1998), 3137–3143.

[6] Y.-H. Lee and K.-W. Jun, *A generalization of the Hyers–Ulam–Rassias stability of Jensen's equation*, J. Math. Anal. Appl., **238**(1999), 305–315.

[7] J. C. Parnami and H. K. Vasudeva, *On Jensen's functional equation*, Aequationes Math., **43**(1992), 211–218.

[8] S. Rolewicz, *Metric linear spaces*, Polish Sci. Publ. Warsaw. (1972).

[9] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297–300.

[10] Th. M. Rassias, *On a modified Hyers–Ulam sequence*, J. Math. Anal. Appl., **158**(1991), 106–113.

[11] Th. M. Rassias and P. Šemrl, *On the Hyers–Ulam stability of linear mappings*, J. Math. Anal. Appl., **173**(1993), 325–338.

[12] Th. M. Rassias and P. Šemrl, *On the behavior of mappings which do not satisfy Hyers–Ulam stability*, Proc. Amer. Math. Soc. **114**(1992), 989–993.

[13] T. Trif, *Hyers–Ulam–Rassias Stability of a Jensen type functional equation*, J. Math. Anal. Appl., **250**(2000), 579–588.

[14] S. M. Ulam, *Problems in modern mathematics*, Chapter VI, Science Editions, Wiley, New York, 1964.

[15] J. Wang, *Some Further Generalizations of the Hyers–Ulam–Rassias Stability of Functional Equations*, J. Math. Anal. Appl., **263** (2004), 406–423.

[16] J. Wang, *On the Generalizations of the Hyers–Ulam–Rassias Stability of Cauc-hy Equations*, Acta Analysis. Funct. Appl. **4**(2002), 294–300.

[17] J. Wang, *On the Generalizations of the Hyers-Ulam Stability of Pexider Equations and Jensen Equations,* Nonlinear Funct. Anal. Appl., **7**(2002), 229-239.

[18] J. Wang, *The additive approximation on a four-variable Jensen type operator equation,* Internat. J. Math. Math. Sci., Wbf 50, (2001), 3171-3187.

[19] A. Wilansky, *Modern methods in topological vector spaces,* McGraw-Hill, New York, 1978.