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MULTIPLE POSITIVE SOLUTIONS FOR SINGULAR THREE-POINT BOUNDARY VALUE PROBLEM SYSTEM

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Abstract. Some results about the existence of multiple positive solutions for a singular three-point boundary value problem system are obtained by using the fixed point index.

1. Introduction

In this paper, we study the existence of multiple positive solutions of the three-point boundary value problem system

$$\begin{cases} x'' + \lambda a_1(t) f_1(x(t), y(t)) = 0, & 0 < t < 1, \\ y'' + \lambda a_2(t) f_2(x(t), y(t)) = 0, & 0 < t < 1, \\ x(0) = 0 = x(1) - \alpha_1 x(\eta_1), \\ y(0) = 0 = y(1) - \alpha_2 y(\eta_2), \end{cases} \tag{1.1_λ}$$

where $\lambda > 0$ is a parameter, $a_1, a_2 : (0,1) \mapsto (0,\infty)$ are continuous, $f_1 : (0,\infty) \times [0,\infty) \mapsto (0,\infty)$ is continuous, $f_2 : [0,\infty) \times (0,\infty) \mapsto (0,\infty)$ is continuous, $\alpha_1, \alpha_2 \in [0,1)$, $\eta_1, \eta_2 \in (0,1)$.

The multi-point boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics. During the last ten years finding solutions, especially positive solutions, to the multi-point boundary value problems has been actively pursued and significant progress has taken place, see [1,4,6,11,12,13,14,16] and the references therein.

In recent years, there were some papers considered the existence of solutions for the two-point boundary value problem system, see [3,8,9] and the references therein. For example, Agarwal and O'Regan [9] studied the two-point

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boundary value problem system

$$\begin{cases} u'' + f(t, v(t)) = 0, & \text{a.e. } t \in [0,1], \\ v'' + g(t, u(t)) = 0, & \text{a.e. } t \in [0,1], \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \ \gamma_1 u(1) + \delta_1 u'(1) = 0, & \\ \alpha_2 v(0) - \beta_2 v'(0) = 0, \ \gamma_2 v(1) + \delta_2 v'(1) = 0. & \end{cases} \tag{1.2}$$

By using Leray-Schauder theory, Agarwal and O'Regan obtained some existence results for solutions for the two-point boundary value problems system (1.2).

However, to the author's knowledge, there is fewer papers considered the existence of multiple positive solutions for multi-point boundary value problems system. To cover up this gap, by using the fixed point index method we will give some existence results for multiple positive solutions for the system (1.1_λ). The system (1.1_λ) is a singular boundary value problem system since the nonlinearity $a_1(t)f_1(x, y)$ is allowed to have singularity at $t = 0, 1$ and $x = 0$, and $a_2(t)f_2(x, y)$ is allowed to have singularity at $t = 0, 1$ and $y = 0$. Singular differential boundary value problems arise in the fields of gas dynamics, Newtonian fluid mechanics, nuclear physics, the theory of boundary layer, nonlinear optics and so on. The readers may refer [2,5,7,15] for some recent results on singular differential boundary value problems.

Lastly, we should point out that, the system (1.1_λ) can arise from the study of positive radial solutions of some elliptic boundary value problems system in an annulus $R_1 \le |x| \le R_2$ in R^N, $N \ge 2$, namely, the following systems:

$$\begin{cases} \Delta u + \lambda a_1(|x|) f_1(u, v) = 0, & R_1 < |x| < R_2, \\ \Delta v + \lambda a_2(|x|) f_2(u, v) = 0, & R_1 < |x| < R_2, \\ u(R_1) = v(R_1) = 0, & \\ u(R_2) - \alpha_1 u(R_3) = v(R_2) - \alpha_2 v(R_4) = 0, & \end{cases} \tag{1.3_λ}$$

where $R_1 < R_3 < R_2$, $R_1 < R_4 < R_2$. Therefore, the results of this paper can apply to the study the existence of positive solutions of the system (1.3_λ). To the author's knowledge, no one has studied the existence of positive solutions for the systems (1.3_λ).

2. Several lemmas

Let us list some conditions to be used in this paper.

(H_1) $$\gamma = \int_0^1 s(1-s)(a_1(s) + a_2(s))ds < \infty.$$

(H_2) $$f_1(x, y) = g_1(x) + h_1(x, y), \ (x, y) \in (0, \infty) \times [0, \infty),$$

$$f_2(x, y) = g_2(y) + h_2(x, y), \ (x, y) \in [0, \infty) \times (0, \infty),$$

where $g_1, g_2 \in C((0,\infty),(0,\infty))$ are non-increasing, $h_1, h_2 \in C([0,\infty)\times[0,\infty),[0,\infty))$, and there exists $R_0 > 0$ such that

$$A(R_0) \triangleq \inf_{(x,y)\in D_{R_0}} \Big(\int_{\alpha_1 x}^{x} \frac{ds}{g_1(s)} + \int_{\alpha_2 y}^{y} \frac{ds}{g_2(s)}\Big) > 0,$$

where $D_{R_0} = \{(x,y) \in [0,\infty)\times[0,\infty) | x+y = R_0\}$.

Let $E = C[0,1]$ be the usual real Banach space with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$, and $P = \{x \in E | x(t) \geq 0 \text{ for } t \in [0,1]\}$. Then P is a cone of E. Let $E^{\Delta} = E \times E$. For any $(x,y) \in E^{\Delta}$, let

$$\|(x,y)\| = \|x\| + \|y\|.$$

Then E^{Δ} is a real Banach space with the norm $\|(\cdot,\cdot)\|$. For $i = 1,2$, let

$$Q_i = \{x \in P | x(t) \geq \|x\| e_i(t) \text{ for } t \in [0,1]\},$$

where

$$e_i(t) = \begin{cases} \dfrac{1-\eta_i}{2-\alpha_i\eta_i-\alpha_i} t, & t \in [0,\eta_i], \\ \dfrac{\eta_i[(1-\alpha_i\eta_i)-(1-\alpha_i)t]}{2-\alpha_i\eta_i-\alpha_i}, & t \in [\eta_i,1]. \end{cases}$$

Then Q_1 and Q_2 are cones of E, and $Q^{\Delta} = Q_1 \times Q_2$ is a cone of E^{Δ}.

Now for i=1,2, let us define the linear operator $K_i : P \mapsto P$ by

$$(K_i x)(t) = \begin{cases} \dfrac{1}{1-\alpha_i\eta_i} \displaystyle\int_0^1 G_{[0,1]}(\eta_i,s) a_i(s) x(s) ds, \ t = \eta_i, \\ \displaystyle\int_0^{\eta_i} G_{[0,\eta_i]}(t,s) a_i(s) x(s) ds + (K_i x)(\eta_i) \frac{t}{\eta_i}, \ t \in [0,\eta_i], \\ \displaystyle\int_{\eta_i}^1 G_{[\eta_i,1]}(t,s) a_i(s) x(s) ds + (K_i x)(\eta_i) \frac{1-t+\alpha_i(t-\eta_i)}{1-\eta_i}, \\ \qquad t \in [\eta_i,1], \end{cases}$$

where

$$G_{[0,1]}(t,s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_{[0,\eta_i]}(t,s) = \begin{cases} (1-\eta_i^{-1}t)s, & 0 \leq s \leq t \leq \eta_i, \\ t(1-\eta_i^{-1}s), & 0 \leq t \leq s \leq \eta_i, \end{cases}$$

$$G_{[\eta_i,1]}(t,s) = \begin{cases} (1-\eta_i)^{-1}(1-t)(s-\eta_i), & \eta_i \leq s \leq t \leq 1, \\ (1-\eta_i)^{-1}(t-\eta_i)(1-s), & \eta_i \leq t \leq s \leq 1. \end{cases}$$

Let N be the set of all positive integers. For each $n \in N$ and $i = 1,2$, let us define the operators $F_{in} : P^{\Delta} \mapsto P$, $A_{in} : P^{\Delta} \mapsto P$and $A_n : P^{\Delta} \mapsto P^{\Delta}$ by

$$F_{1n}(x,y)(t) = f_1(x(t)+n^{-1}, y(t)), t \in [0,1], \ (x,y) \in P^{\Delta},$$

$$F_{2n}(x,y)(t) = f_2(x(t), y(t)+n^{-1}), t \in [0,1], (x,y) \in P^{\Delta},$$

$$A_{1n}(x,y) = K_1F_{1n}(x,y), A_{2n}(x,y) = K_2F_{2n}(x,y), (x,y) \in P^{\Delta},$$

and

$$A_n(x,y) = (A_{1n}(x,y), A_{2n}(x,y)), (x,y) \in P^{\Delta}, \tag{2.1}$$

respectively.

For each $R > 0$ and $i = 1, 2$, let

$$H_i(R) = \sup\{h_i(x,y) | 0 \le x \le R+1, 0 \le y \le R+1\},$$

$$\rho_i(R) = 1 + \frac{H_i(R)}{g_i(R+1)}.$$

In the same way as the proof of Lemma 8 of [15], we have the following Lemma 2.1.

Lemma 2.1. *Suppose that* (H_1) *holds. Then*

$$\lim_{t\to 0^+} t\int_t^1 (1-s)\sum_{i=1}^2 a_i(s)ds = 0$$

and

$$\lim_{t\to 1^-} (1-t)\int_0^t s\sum_{i=1}^2 a_i(s)ds = 0.$$

By direct computation, we have the following Lemma 2.2.

Lemma 2.2. *Suppose that* (H_1) *holds, and* $h \in P$. *Then for i=1,2,* $\omega_i(t)$ *is a solution of the three-point boundary value problem*

$$\begin{cases} \omega_i'' + a_i(t)h(t) = 0, & 0 < t < 1, \\ \omega_i(0) = 0 = \omega_i(1) - \alpha_i\omega_i(\eta_i) \end{cases}$$

if and only if $\omega_i(t) = (K_ih)(t)$ *for* $t \in [0,1]$.

Remark 2.1. For the proof of Lemma 2.2, the readers can also refer Theorem 5 and 7 of [16].

Lemma 2.3. *Suppose that* (H_1) *holds. Then* $K_i : P \mapsto Q_i$ *is a completely continuous operator for* $i = 1, 2$.

Proof. We only show that $K_1 : P \mapsto Q_1$ is a completely continuous operator. In the same way, we can show that $K_2 : P \mapsto Q_2$ is a completely continuous operator. For any $x \in P$, let $y(t) = (K_1x)(t)$ for $t \in [0,1]$. It follows from Lemma 2.2 that $y \in C[0,1]$ satisfies

$$\begin{cases} y''(t) + a_1(t)x(t) = 0, & 0 < t < 1, \\ y(0) = 0 = y(1) - \alpha_1 y(\eta_1). \end{cases}$$

Thus, y is a concave function on [0,1]. Since the graph of the function y passes through three points $(0,0), (\eta_1, y(\eta_1))$ and $(1, \alpha_1 y(\eta_1))$, then

$$\begin{aligned} y(t) \quad &\leq \begin{cases} (-\frac{1-\alpha_1}{1-\eta_1}t + \frac{1-\alpha_1\eta_1}{1-\eta_1})y(\eta_1), & t \in [0,\eta_1] \\ \frac{t}{\eta_1}y(\eta_1), & t \in [\eta_1, 1] \end{cases} \\ &\leq \begin{cases} \frac{1-\alpha_1\eta_1 + 1 - \alpha_1}{1-\eta_1}y(\eta_1), & t \in [0,\eta_1] \\ \frac{1}{\eta_1}y(\eta_1), & t \in [\eta_1, 1] \end{cases} \end{aligned}$$

and so

$$y(\eta_1) \geq \frac{\eta_1(1-\eta_1)}{2-\alpha_1\eta_1-\alpha_1}\|y\|.$$

Since y is a concave function, we have for $t \in [0, \eta_1]$,

$$\begin{aligned} y(t) \quad &= y(\frac{t}{\eta_1}\eta_1 + \frac{\eta_1 - t}{\eta_1} \cdot 0) \\ &\geq \frac{t}{\eta_1}y(\eta_1) + \frac{\eta_1 - t}{\eta_1}y(0) \\ &\geq \frac{t}{\eta_1}y(\eta_1) \\ &\geq \frac{1-\eta_1}{2-\alpha_1\eta_1-\alpha_1}\|y\|t \end{aligned} \tag{2.2}$$

and for $t \in [\eta_1, 1]$,

$$\begin{aligned} y(t) \quad &= y(\frac{t-\eta_1}{1-\eta_1} \cdot 1 + \frac{1-t}{1-\eta_1} \cdot \eta_1) \\ &\geq \frac{t-\eta_1}{1-\eta_1}y(1) + \frac{1-t}{1-\eta_1}y(\eta_1) \\ &= \frac{1-\alpha_1\eta_1 - (1-\alpha_1)t}{1-\eta_1}y(\eta_1) \\ &\geq \frac{\eta_1[(1-\alpha_1\eta_1) - (1-\alpha_1)t]}{2-\alpha_1\eta_1-\alpha_1}\|y\|. \end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we see that $K_1 : P \mapsto Q_1$.

Now we shall show that K_1 is a completely continuous operator. The continuity and the boundedness of K_1 can be easily obtained. Let $\Omega \subset P$ be a bounded set of P and L a positive number such that $\|x\| \leq L$ and $\|K_1 x\| \leq L$ for all $x \in \Omega$. For any $\varepsilon > 0$, by (H_1) there exists $\delta_1 > 0$ such that

$$\int_0^{\delta_1} G_{[0,\eta_1]}(s,s)a_1(s)ds < \varepsilon. \tag{2.4}$$

Since $G_{[0,\eta_1]}(t,s)$ is uniformly continuous on $[0,\eta_1] \times [0,\eta_1]$, then there exists $0 \leq \delta < \varepsilon$ such that for any $t_1, t_2 \in [0,\eta_1)$ with $|t_1 - t_2| \leq \delta$

$$\int_{\delta_1}^{\eta_1} |G_{[0,\eta_1]}(t_1,s) - G_{[0,\eta_1]}(t_2,s)|a_1(s)ds < \varepsilon. \tag{2.5}$$

Using the fact that $G_{[0,\eta_1]}(t,s) \leq G_{[0,\eta_1]}(s,s)$ for $(t,s) \in [0,\eta_1] \times [0,\eta_1]$, by (2.4) and (2.5) we have for $t_1, t_2 \in [0,\eta_1]$ with $|t_1 - t_2| < \delta$,

$$\begin{aligned} |(K_1x)(t_2) - (K_1x)(t_1)| &\leq (K_1x)(\eta_1)\eta_1^{-1}|t_2 - t_1| \\ &\quad + L\int_0^{\eta_1} |G_{[0,\eta_1]}(t_1,s) - G_{[0,\eta_1]}(t_2,s)|a_1(s)ds \\ &\leq L\eta_1^{-1}|t_2 - t_1| + 2L\int_0^{\delta_1} G_{[0,\eta_1]}(s,s)a_1(s)ds \\ &\quad + L\int_{\delta_1}^{\eta_1} |G_{[0,\eta_1]}(t_1,s) - G_{[0,\eta_1]}(t_2,s)|a_1(s)ds \\ &\leq L(\eta_1^{-1} + 3)\varepsilon \end{aligned}$$

Thus, $K_1(\Omega)$ is equicontinuous on $[0,\eta_1]$. Similarly, $K_1(\Omega)$ is equicontinuous on $[\eta_1, 1]$. According to Ascoli-Arzel*à* Theorem, $K_1(\Omega) \subset C[0,1]$ is a relatively compact set, and so $K_1 : P \mapsto Q_1$ is a completely continuous operator. The proof is completed. □

By Lemma 2.3, we have the following Lemma 2.4.

Lemma 2.4. *Suppose that* (H_1) *holds. Then* $A_n : P^{\Delta} \mapsto Q^{\Delta}$ *is a completely continuous operator for each positive integer* n.

Lemma 2.5. *Suppose that* (H_1) *and* (H_2) *hold,* $\lambda > 0$, $R_\lambda > 0$. *Moreover, for each* $n \in N$, λA_n *has at least one fixed point* (x_n, y_n) *such that* $\|(x_n, y_n)\| \leq R_\lambda$. *Then the set* $\{(x_n, y_n) | n \in N\}$ *is a relatively compact set.*

Proof. Let $z_0(t) = 1$ for $t \in [0,1]$. Then we have for every positive integer n,

$$x_n(t) = \lambda K_1 F_{1n}(x_n, y_n)(t) \geq \lambda g_1(R_\lambda + 1)(K_1 z_0)(t) =: z_{\lambda 1}(t), t \in [0,1], \tag{2.6}$$

$$y_n(t) = \lambda K_2 F_{2n}(x_n, y_n)(t) \geq \lambda g_2(R_\lambda + 1)(K_2 z_0)(t) =: z_{\lambda 2}(t), t \in [0,1]. \tag{2.7}$$

Let us define the function F_1 by

$$F_1(t) = \int_t^1 (1-s)a_1(s)ds, t \in (0,1].$$

Obviously, $F_1 \in C(0,1]$, $F_1(1) = 0$, and $F_1$is non-increasing on $(0,1]$. For each positive integer n, by Lemma 2.2, we have

$$\begin{cases} -x_n''(t) = \lambda a_1(t) f_1(x_n(t) + n^{-1}, y_n(t)), & 0 < t < 1, \\ x_n(0) = 0 = x_n(1) - \alpha_1 x_n(\eta_1) \end{cases} \tag{2.8}$$

and

$$\begin{cases} -y_n''(t) = \lambda a_2(t) f_2(x_n(t), y_n(t) + n^{-1}), & 0 < t < 1, \\ y_n(0) = 0 = y_n(1) - \alpha_2 y_n(\eta_2) \end{cases} \tag{2.9}$$

By (2.8), we see that x_n is a strictly concave function on $(0,1)$. Then there exists unique $t_{n1} \in (0,1)$ such that $x_n'(t_{n1}) = 0$ and $x_n(t_{n1}) = \|x_n\|$. By (H_2) and (2.8), we have

$$-x_n''(t) \leq \lambda a_1(t) g_1(x_n(t)) \rho_1(R_\lambda),\ t \in (0,1).$$

Integrate from t_{n1} to $t (t \in (t_{n1}, 1))$ to obtain

$$\frac{-x_n'(t)}{g_1(x_n(t))} \leq \lambda \rho_1(R_\lambda) \int_{t_{n1}}^{t} a_1(s) ds. \tag{2.10}$$

Then integrate from t_{n1} to 1 to obtain

$$\int_{x_n(1)}^{x_n(t_{n1})} \frac{ds}{g_1(s)} \leq \lambda \rho_1(r_\lambda) \int_{t_{n1}}^{1} (1-s) a_1(s) ds = \lambda \rho_1(R_\lambda) F_1(t_{n1}) \tag{2.11}$$

On the other hand, by (2.6), we have when $\alpha_1 \in (0,1)$,

$$\begin{aligned} \int_{x_n(1)}^{x_n(t_{n1})} \frac{ds}{g_1(s)} &\geq \frac{x_n(t_{n1}) - x_n(1)}{g_1(x_n(1))} \\ &= \frac{x_n(t_{n1}) - \alpha_1 x_n(\eta_1)}{g_1(x_n(1))} \\ &\geq \frac{x_n(\eta_1)(1-\alpha_1)}{g_1(x_n(1))} \\ &\geq \frac{z_{\lambda 1}(\eta_1)(1-\alpha_1)}{g_1(z_{\lambda 1}(1))} > 0 \end{aligned} \tag{2.12}$$

and when $\alpha_1 = 0$,

$$\int_{x_n(1)}^{x_n(t_{n1})} \frac{ds}{g_1(s)} \geq \int_0^{\|z_{\lambda 1}\|} \frac{ds}{g_1(s)}. \tag{2.13}$$

By (2.11)-(2.13), we have

$$F_1(t_{n1}) \geq [\lambda \rho_1(R_\lambda)]^{-1} \min\{\frac{z_{\lambda 1}(\eta_1)(1-\alpha_1)}{g_1(z_{\lambda 1}(1))}, \int_0^{\|z_{\lambda 1}\|} \frac{ds}{g_1(s)}\}. \tag{2.14}$$

Let $\beta_{01} \in (0,1)$ be such that

$$F_1(\beta_{01}) = [\lambda \rho_1(R_\lambda)]^{-1} \min\{\frac{z_{\lambda 1}(\eta_1)(1-\alpha_1)}{g_1(z_{\lambda 1}(1))}, \int_0^{\|z_{\lambda 1}\|} \frac{ds}{g_1(s)}\}.$$

Then (2.14) implies that $t_{n1} \le \beta_{01}$. Similarly, there exists $\alpha_{01} \in (0,1)$ such that $t_{n1} \ge \alpha_{01}$.

Now let us define the function $I : [0,\infty) \mapsto [0,\infty)$ by $I(x) = \displaystyle\int_0^x \frac{ds}{g_1(s)}$ for $x \in [0,\infty)$. For any $t_1, t_2 \in [\beta_{01}, 1]$, $t_1 \le t_2$, by (2.10), we have

$$\begin{aligned} I(x_n(t_1)) - I(x_n(t_2)) &= \int_{x_n(t_2)}^{x_n(t_1)} \frac{ds}{g_1(s)} = \int_{t_1}^{t_2} \frac{-x_n'(s)ds}{g_1(x_n(s))} \\ &\le \lambda\rho_1(R_\lambda) \int_{t_1}^{t_2} ds \int_{\alpha_{01}}^{s} a_1(\tau)d\tau \\ &\le \lambda\rho_1(R_\lambda)(\int_{t_1}^{t_2} (t_2 - s)a_1(s)ds \\ &+(t_2 - t_1)\int_{\alpha_{01}}^{t_1} a_1(s)ds) \\ &\le \lambda\rho_1(R_\lambda)(\int_{t_1}^{t_2} (1 - s)a_1(s)ds \\ &+\frac{1}{\alpha_{01}}(t_2 - t_1)\int_{\alpha_{01}}^{1-(t_2-t_1)} sa_1(s)ds) \end{aligned}$$

This and the inequalities of Lemma 2.1 imply that the set $I(\{x_n|n \in N\})$ is euicontinuous on $[\beta_{01}, 1]$. It is easy to see that I^{-1}, the converse function of I, is uniformly continuous on $[0, I(R_\lambda)]$. Therefore, the set $\{x_n|n \in N\}$ is euicontinuous on $[\beta_{01}, 1]$. Similarly, $\{x_n|n \in N\}$ is equicontinuous on $[0, \alpha_{01}]$. From (2.6), we have for $t \in [\alpha_{01}, \beta_{01}]$

$$|x_n'(t)| \le \lambda(g_1(\min_{t\in[\alpha_{01},\beta_{01}]} z_{\lambda 1}(t)) + H_1(R_\lambda)) \int_{\alpha_{01}}^{\beta_{01}} a_1(s)ds.$$

Thus $\{x_n|n \in N\}$ is equicontinuous on $[\alpha_{01}, \beta_{01}]$. Then, by Ascoli-*Arzelà* Theorem, we see that $\{x_n|n \in N\} \subset C[0,1]$ is a relatively compact set.

Similarly, by (2.7) and (2.9), we can show that $\{y_n|n \in N\} \subset C[0,1]$ is also a relatively compact set. Therefore, $\{(x_n, y_n)|n \in N\}$ is a relatively compact set. The proof is completed. □

3. Main results

Theorem 3.1. *Suppose that* (H_1) *and* (H_2) *hold. Moreover,*

$$\lim_{y\to\infty} \frac{h_1(x,y)}{y} = \infty \textit{ uniformly respect to } \ x \in [0,\infty),$$

$$\lim_{x\to\infty} \frac{h_2(x,y)}{x} \ge a \textit{ uniformly respect to } \ y \in [0,\infty),$$

where a is a positive number. Then there exists $\lambda^* > 0$ such that the system (1.1_λ) has at least two positive solutions for $0 < \lambda < \lambda^*$.

Proof. Let the operator A_n be defined by (2.1) for each positive integer n. Take $0 < R_0' < R_0$ be such that

$$A(R_0') \triangleq \inf_{(x,y)\in D_{R_0'}} \left(\int_{\alpha_1 x}^{x} \frac{ds}{g_1(s)} + \int_{\alpha_2 y}^{y} \frac{ds}{g_2(s)} \right) > 0.$$

Let the positive number λ^* be such that

$$\lambda^* < \min\left\{ \frac{A(R_0)}{2[\rho_1(R_0) + \rho_2(R_0)]\gamma}, \frac{A(R_0')}{2[\rho_1(R_0') + \rho_2(R_0')]\gamma} \right\}.$$

Let $\lambda_0 \in (0, \lambda^*)$ be fixed. Now we shall show that

$$(x,y) \neq \mu\lambda_0 A_n(x,y), \forall n \in N, (x,y) \in \partial B_{R_0} \cap Q^\Delta, \mu \in [0,1], \tag{3.1}$$

where $B_{R_0} = \{(x,y) \in P^\Delta | \|(x,y)\| < R_0\}$. In fact, if not, then there exist $n_0 \in N, \mu_0 \in (0,1]$ and $(x_0, y_0) \in \partial B_{R_0} \cap Q^\Delta$ such that

$$(x_0, y_0) = \mu_0\lambda_0 A_{n_0}(x_0, y_0). \tag{3.2}$$

From (H_2), we see that

$$x_0(\eta_1) = \mu_0\lambda_0 K_1 F_{1n_0}(x_0, y_0)(\eta_1) \geq \mu_0\lambda_0 g_1(R_0 + 1)(K_1 z_0)(\eta_1) > 0,$$

$$y_0(\eta_2) = \mu_0\lambda_0 K_2 F_{2n_0}(x_0, y_0)(\eta_2) \geq \mu_0\lambda_0 g_2(R_0 + 1)(K_2 z_0)(\eta_2) > 0,$$

where $z_0(t) = 1$ for $t \in [0,1]$. Then we have for $t \in [0,1]$,

$$x_0(t) \geq \|x\| e_1(t) > 0, \quad y_0(t) \geq \|y\| e_2(t) > 0. \tag{3.3}$$

By (3.2) and Lemma 2.2, we have

$$\begin{cases} x_0''(t) + \mu_0\lambda_0 a_1(t) f_1(x_0(t) + n_0^{-1}, y_0(t)) = 0, & 0 < t < 1, \\ x_0(0) = 0 = x_0(1) - \alpha_1 x_0(\eta_1) \end{cases} \tag{3.4}$$

and

$$\begin{cases} y_0''(t) + \mu_0\lambda_0 a_2(t) f_2(x_0(t), y_0(t) + n_0^{-1}) = 0, & 0 < t < 1, \\ y_0(0) = 0 = y_0(1) - \alpha_2 y_0(\eta_2) \end{cases} \tag{3.5}$$

By (3.4), we see that x_0 is a strictly concave function on $[0,1]$, and there exists $t_1 \in (0,1)$ such that $x_0'(t_1) = 0, x_0(t_1) = \|x_0\|$. By (3.4), and (H_2), we have

$$-x_0''(t) \leq \lambda_0 a_1(t) g_1(x_0(t)) \rho_1(R_0), \ t \in (0,1). \tag{3.6}$$

Integrate from $t (t \in (0, t_1))$ to t_1 to obtain

$$\frac{x_0'(t)}{g_1(x_0(t))} \leq \lambda_0 \rho_1(R_0) \int_t^{t_1} a_1(s) ds$$

Then integrate from 0 to t_1 to obtain

$$\int_{x_0(0)}^{x_0(t_1)} \frac{ds}{g_1(s)} \leq \lambda_0\rho_1(R_0)\int_0^{t_1} sa_1(s)ds \leq \frac{\lambda_0\rho_1(R_0)\gamma}{1-t_1}. \tag{3.7}$$

In the same way as the proof of (2.11), by (3.6), we can show that

$$\int_{x_0(1)}^{x_0(t_1)} \frac{ds}{g_1(s)} \leq \lambda_0\rho_1(R_0)\int_{t_1}^{1} (1-s)a_1(s)ds \leq \frac{\lambda_0\rho_1(R_0)\gamma}{t_1}. \tag{3.8}$$

By (3.7) and (3.8), we have

$$\int_{\alpha_1\|x_0\|}^{\|x_0\|} \frac{ds}{g_1(s)} \leq \int_{x_0(1)}^{x_0(t_1)} \frac{ds}{g_1(s)} \leq 2\lambda_0\rho_1(R_0)\gamma \tag{3.9}$$

By (3.5), we can show that

$$\int_{\alpha_2\|y_0\|}^{\|y_0\|} \frac{ds}{g_2(s)} \leq 2\lambda_0\rho_2(R_0)\gamma. \tag{3.10}$$

By (3.9) and (3.10), we have

$$A(R_0) \leq \int_{\alpha_1\|x_0\|}^{\|x_0\|} \frac{ds}{g_1(s)} + \int_{\alpha_2\|y_0\|}^{\|y_0\|} \frac{ds}{g_2(s)} \leq 2\lambda_0(\rho_1(R_0)+\rho_2(R_0))\gamma$$

and so

$$\lambda_0 \geq \frac{A(R_0)}{2\gamma(\rho_1(R_0)+\rho_2(R_0))},$$

which is a contradiction. This implies that (3.1) holds. Then we have for each positive integer n,

$$i(\lambda_0 A_n, B_{R_0}\cap Q^{\Delta}, Q^{\Delta}) = 1. \tag{3.11}$$

Similarly, we can show that

$$i(\lambda_0 A_n, B_{R_0'}\cap Q^{\Delta}, Q^{\Delta}) = 1. \tag{3.12}$$

Let $[\alpha,\beta]\subset(0,1)$, and

$$\gamma_1 = \int_{\alpha}^{\beta} G_{[0,1]}(\eta_1,s)a_1(s)e_2(s)ds,$$

$$\gamma_2 = \int_{\alpha}^{\beta} G_{[0,1]}(\eta_2,s)a_2(s)e_1(s)ds.$$

Then, for $M_{\lambda_0} > \max\{(\lambda_0\gamma_1)^{-1}, 4(\lambda_0^2\gamma_1\gamma_2 a)^{-1}\}$, there exists $R'_{\lambda_0}>0$ such that

$$h_1(x,y) \geq M_{\lambda_0}y, y\geq R'_{\lambda_0}, x\geq 0 \tag{3.13}$$

and

$$h_2(x,y) \geq \frac{1}{2}ax, x\geq R'_{\lambda_0}, y\geq 0. \tag{3.14}$$

Let $c_0 = \min\{\min\limits_{t\in[\alpha,\beta]} e_1(t), \min\limits_{t\in[\alpha,\beta]} e_2(t)\}$, $n_1 > c_0^{-1}$ and

$$R_{\lambda_0} \geq \max\{4n_1R'_{\lambda_0}(\lambda_0 a\gamma_2)^{-1}, 2R'_{\lambda_0}c_0^{-1}, 4n_1R'_{\lambda_0}, R_0\}.$$

Let $(\Psi_1,\Psi_2) \in Q^{\Delta}\backslash\{\theta,\theta\}$. Now we shall show that

$$(x,y) \neq \lambda_0 A_n(x,y) + \mu(\Psi_1,\Psi_2), (x,y) \in \partial B_{R_{\lambda_0}} \cap Q^{\Delta}, \mu \geq 0, n \in N. \quad (3.15)$$

In fact, if not, assume that for some $(x_0,y_0) \in \partial B_{R_{\lambda_0}} \cap Q^{\Delta}$, $n_0 \in N$ and $\mu_0 \geq 0$,

$$(x_0,y_0) = \lambda_0 A_{n_0}(x_0,y_0) + \mu_0(\Psi_1,\Psi_2).$$

Since $\|(x_0,y_0)\| = R_{\lambda_0}$, then we have one of these two cases:

(1) $\|x_0\| \geq \dfrac{R_{\lambda_0}}{2}$. Since $x_0 \in Q_1$, then we have

$$x_0(t) \geq \|x_0\|e_1(t) \geq \frac{R_{\lambda_0}}{2}\min_{t\in[\alpha,\beta]} e_1(t) \geq R'_{\lambda_0}, t \in [\alpha,\beta].$$

By (3.14), we have

$$\begin{aligned}
\|y_0\| &\geq y_0(\eta_2) = \lambda_0\int_0^1 G_{[0,1]}(\eta_2,s)a_2(s)f_2(x_0(s),y_0(s)+n_0^{-1})ds + \mu_0\Psi_2(\eta_2)\\
&\geq \lambda_0\int_0^1 G_{[0,1]}(\eta_2,s)a_2(s)f_2(x_0(s),y_0(s)+n_0^{-1})ds\\
&\geq \frac{1}{2}\lambda_0 a\int_\alpha^\beta G_{[0,1]}(\eta_2,s)a_2(s)x_0(s)ds\\
&\geq \frac{1}{2}\|x_0\|\lambda_0 a\gamma_2\\
&\geq \frac{1}{4}R_{\lambda_0}\lambda_0 a\gamma_2\\
&\geq n_1R'_{\lambda_0}
\end{aligned}$$

(2) $\|y_0\| \geq \dfrac{R_{\lambda_0}}{2}$. Since $y_0 \in Q_2$, then we have

$$y_0(t) \geq \|y_0\|e_2(t) \geq \frac{R_{\lambda_0}}{2}\min_{t\in[\alpha,\beta]} e_2(t) \geq R'_{\lambda_0}, t \in [\alpha,\beta].$$

By (3.13), we have

$$\begin{aligned}\|x_0\| &\geq x_0(\eta_1) = \lambda_0 \int_0^1 G_{[0,1]}(\eta_1, s)a_1(s)f_1(x_0(s) + n_0^{-1}, y_0(s))ds + \mu_0\Psi_1(\eta_1)\\ &\geq \lambda_0 \int_0^1 G_{[0,1]}(\eta_1, s)a_1(s)f_1(x_0(s) + n_0^{-1}, y_0(s))ds\\ &\geq \frac{1}{2}\lambda_0 M_{\lambda_0}\|y_0\|\gamma_1\\ &\geq \frac{R_{\lambda_0}}{4}\lambda_0 M_{\lambda_0}\gamma_1\\ &\geq \frac{R_{\lambda_0}}{4}\\ &\geq n_1 R'_{\lambda_0}\end{aligned}$$

Now, from the arguments of (1) and (2), we have

$$x_0(t) \geq \|x_0\|e_1(t) \geq n_1 R'_{\lambda_0} \min_{t\in[\alpha,\beta]} e_1(t) \geq R'_{\lambda_0}, t \in [\alpha, \beta],$$

$$y_0(t) \geq \|y_0\|e_2(t) \geq n_1 R'_{\lambda_0} \min_{t\in[\alpha,\beta]} e_2(t) \geq R'_{\lambda_0}, t \in [\alpha, \beta].$$

Then, by (3.13) and (3.14), we have

$$\begin{aligned}\|x_0\| &\geq x_0(\eta_1) \geq \lambda_0 \int_\alpha^\beta G_{[0,1]}(\eta_1, s)a_1(s)h_1(x_0(s) + n_0^{-1}, y_0(s))ds\\ &\geq \lambda_0 M_{\lambda_0}\|y_0\|\gamma_1\end{aligned} \tag{3.16}$$

$$\begin{aligned}\|y_0\| &\geq y_0(\eta_2) \geq \lambda_0 \int_0^1 G_{[0,1]}(\eta_2, s)a_2(s)f_2(x_0(s), y_0(s) + n_0^{-1})ds\\ &\geq \lambda_0 \int_\alpha^\beta G_{[0,1]}(\eta_2, s)a_2(s)h_2(x_0(s), y_0(s) + n_0^{-1})ds\\ &\geq \frac{1}{2}\lambda_0 a\|x_0\|\gamma_2\end{aligned} \tag{3.17}$$

From (3.16) and (3.17), we have

$$\|x_0\| \geq \lambda_0 M_{\lambda_0}\gamma_1 \cdot \frac{1}{2}\lambda_0 a\gamma_2\|x_0\| \geq 2\|x_0\|,$$

which is a contradiction. This implies that (3.15) holds. From the properties of the fixed point index, we have

$$i(\lambda_0 A_n, B_{R_{\lambda_0}} \cap Q^\Delta, Q^\Delta) = 0, \forall n \in N. \tag{3.18}$$

It follows from (3.11) and (3.18) that

$$i(\lambda_0 A_n, (B_{R_\lambda}\backslash\bar{B}_{R_0}) \cap Q^\Delta, Q^\Delta) = -1, \forall n \in N.$$

Thus, the operator $\lambda_0 A_n$ has at least one fixed point (x_{n1}, y_{n1}) such that $\|(x_{n1}, y_{n1})\| \leq R_{\lambda_0}$. From Lemma 2.5, $\{(x_{n1}, y_{n1})|n \in N\}$ is a relatively compact set. Without loss of generality, we may assume that $(x_{n1}, y_{n1}) \to (x_1, y_1)$ as $n \to \infty$. By Lemma 2.2, we have for each positive integer n

$$\begin{cases} x''_{n1}(t) + \lambda_0 a_1(t) f_1(x_{n1}(t) + n^{-1}, y_{n1}(t)) = 0, \quad 0 < t < 1, \\ x_{n1}(0) = 0 = x_{n1}(1) - \alpha_1 x_{n1}(\eta_1), \end{cases} \tag{3.19}$$

and

$$\begin{cases} y''_{n1}(t) + \lambda_0 a_2(t) f_2(x_{n1}(t), y_{n1}(t) + n^{-1}) = 0, \quad 0 < t < 1, \\ y_{n1}(0) = 0 = y_{n1}(1) - \alpha_2 y_{n1}(\eta_2), \end{cases} \tag{3.20}$$

By (3.19), we have for $t \in (0, 1)$,

$$\begin{aligned} x_{n1}(t) \ &= x_{n1}(\tfrac{1}{2}) + (t - \tfrac{1}{2}) x'_{n1}(\tfrac{1}{2}) \\ &- \int_{\frac{1}{2}}^{t} ds \int_{\frac{1}{2}}^{s} \lambda_0 a_1(\tau) f_1(x_{n1}(\tau) + n^{-1}, y_{n1}(\tau)) d\tau, \end{aligned}$$

Thus, $\{x'_{n1}(\frac{1}{2}) | n \in N\}$ is a bounded set. Without loss of generality, assume that $x'_{n1}(\dfrac{1}{2}) \to a_1$ as $n \to \infty$. Then, using the Lebesgue dominant Theorem, we have

$$\begin{aligned} x_1(t) \ &= x_1(\frac{1}{2}) + (t - \frac{1}{2}) a_1 \\ &- \int_{\frac{1}{2}}^{t} ds \int_{\frac{1}{2}}^{s} \lambda_0 a_1(\tau) f_1(x_1(\tau), y_1(\tau)) d\tau, t \in (0, 1). \end{aligned}$$

A direct computation shows

$$x''_0(t) + \lambda_0 a_1(t) f_1(x_0(t), y_0(t)) = 0, t \in (0, 1). \tag{3.21}$$

By (3.19), we have

$$x_1(0) = 0 = x_1(1) - \alpha_1 x_1(\eta_1). \tag{3.22}$$

Similarly, by (3.20) we can show that

$$\begin{cases} y''_1(t) + \lambda_0 a_2(t) f_2(x_1(t), y_1(t)) = 0, \quad 0 < t < 1, \\ y_1(0) = 0 = y_1(1) - \alpha_2 y_1(\eta_2) \end{cases} \tag{3.23}$$

By(3.21)-(3.23), we see that (x_1, y_1) is a positive solution of (1.1_{λ_0}).

By (3.12), we see that for every $n \in N$ $\lambda_0 A_n$ has at least one fixed point (x_{n2}, y_{n2}) such that $\|(x_{n2}, y_{n2})\| \leq R'_0$. It follows from Lemma 2.5 that $\{(x_{n2}, y_{n2}) | n \in N\}$ is a relatively compact set. Without loss of generality, assume that $(x_{n2}, y_{n2}) \to (x_2, y_2)$ as $n \to \infty$. In the same way as above, we can show that (x_2, y_2) is also a positive solution of (1.1_{λ_0}). Since $\lambda_0 \in (0, \lambda_*)$ is arbitrarily given, then the conclusion holds. This completes the proof. $\square$

In a similar way, we have the following Theorem 3.2.

Theorem 3.2. *Suppose that* (H_1) *and* (H_2) *hold. Moreover,*

$$\lim_{y\to\infty}\frac{h_1(x,y)}{y}\geq a \text{ uniformly respect to } x\in[0,\infty),$$

$$\lim_{x\to\infty}\frac{h_2(x,y)}{x}=\infty \text{ uniformly respect to } y\in[0,\infty),$$

where a *is a positive number. Then there exists* $\lambda^*>0$ *such that the system* (1.1_λ) *has at least two positive solutions for* $0<\lambda<\lambda^*$.

Example 3.1. Consider the following three-point differential equation systems

$$\begin{cases} x''+\lambda t^{-\frac{5}{3}}(1-t)^{-\frac{3}{7}}(x^{-6}+x^{\frac{1}{2}}+y^{\frac{1}{2}}+y^2)=0, & 0<t<1;\\ y''+\lambda t^{-\frac{1}{3}}(1-t)^{-\frac{5}{7}}(x^{\frac{1}{2}}+2x+y^2+y^{-2})=0, & 0<t<1;\\ x(0)=0=x(1)-\frac{5}{6}x(\frac{2}{3}), \\ y(0)=0=y(1)-\frac{8}{7}x(\frac{2}{7}) \end{cases} \tag{3.24$_\lambda$}$$

Let $g_1(x)=x^{-6}, g_2(y)=y^{-2}$, $a_1(t)=t^{-\frac{5}{3}}(1-t)^{-\frac{3}{7}}, a_2(t)=t^{-\frac{1}{3}}(1-t)^{-\frac{5}{7}}$, $h_1(x,y)=x^{\frac{1}{2}}+y^{\frac{1}{2}}+y^2$, $h_2(x,y)=x^{\frac{1}{2}}+2x+y^2$. Then all conditions of Theorem 3.1 are satisfied. According to Theorem 3.1, the system (3.24_λ) has at least two positive solutions for sufficiently small $\lambda>0$.

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