Nonlinear Functional Analysis and Applications
Vol. 12, No. 4 (2007), pp. 595-605

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

ON SOME DISCRETE INEQUALITIES IN NORMED LINEAR SPACES

Sever S. Dragomir

School of Computer Science and Mathematics, Victoria University,
PO Box 14428, Melbourne City, Victoria 8001, Australia.
e-mail: sever.dragomir@vu.edu.au

Abstract. Some sharp discrete inequalities in normed linear spaces are obtained. New reverses of the generalised triangle inequality are also given.

1. Introduction

Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. The mapping $f : X \to \mathbb{R}$, $f(x) = \frac{1}{2}\|x\|^2$ is obviously convex on X and then there exists the following limits:

$$\langle x, y\rangle_i := \lim_{t\to 0-} \frac{\|y+tx\|^2 - \|y\|^2}{2t},$$
$$\langle x, y\rangle_s := \lim_{\tau\to 0+} \frac{\|y+\tau x\|^2 - \|y\|^2}{2\tau}$$

for any two vectors in X. The mapping $\langle \cdot, \cdot\rangle_s$ $(\langle \cdot, \cdot\rangle_i)$ will be called the *superior semi-inner product* (*inferior semi-inner product*) associated to the norm $\|\cdot\|$.

[0]Received November 9, 2006. Revised December 12, 2006.
[0]2000 Mathematics Subject Classification: 46B05, 26D15.
[0]Keywords: Inequalities in normed spaces, semi-inner products, analytic inequalities.

The following fundamental calculus rules are valid for these semi-inner products (see for instance [4, p. 27–32]):

$$\langle x,x\rangle_p = \|x\|^2 \quad \text{for } x\in X; \tag{1.1}$$
$$\langle \lambda x,y\rangle_p = \lambda\langle x,y\rangle_p \quad \text{for } \lambda\ge 0 \text{ and } x,y\in X; \tag{1.2}$$
$$\langle x,\lambda y\rangle_p = \lambda\langle x,y\rangle_p \quad \text{for } \lambda\ge 0 \text{ and } x,y\in X; \tag{1.3}$$
$$\langle \lambda x,y\rangle_p = \lambda\langle x,y\rangle_q \quad \text{for } \lambda\le 0 \text{ and } x,y\in X; \tag{1.4}$$
$$\langle \alpha x,\beta y\rangle_p = \alpha\beta\langle x,y\rangle_p \quad \text{for } \alpha,\beta\in\mathbb{R} \text{ with } \alpha\beta\ge 0 \text{ and } x,y\in X; \tag{1.5}$$
$$\langle -x,y\rangle_p = \langle x,-y\rangle_p = -\langle x,y\rangle_q \quad \text{for } x,y\in X; \tag{1.6}$$

where $p,q\in\{s,i\}$ and $p\ne q$.

The following inequality is valid:

$$\frac{\|y+tx\|^2-\|y\|^2}{2t} \ge \langle x,y\rangle_s \ge \langle x,y\rangle_i \ge \frac{\|y+sx\|^2-\|y\|^2}{2s}, \tag{1.7}$$

for any $x,y\in X$ and $s<0<t$.

An important result is the following *Schwarz inequality*:

$$\left|\langle x,y\rangle_p\right| \le \|x\|\,\|y\| \quad \text{for each } x,y\in X. \tag{1.8}$$

Also, the following properties of sub(super)-additivity should be noted:

$$\langle x_1+x_2,y\rangle_{s(i)} \le (\ge)\,\langle x_1,y\rangle_{s(i)} + \langle x_2,y\rangle_{s(i)} \tag{1.9}$$

for each $x_1,x_2,y\in X$.

Another important property of "quasi-linearity" holds as well:

$$\langle \alpha x+y,x\rangle_p = \alpha\|x\|^2 + \langle y,x\rangle_p \tag{1.10}$$

for any $x,y\in X$ and α a real number, where $p=s$ or $p=i$.

Finally, we mention the continuity property:

$$\left|\langle y+z,x\rangle_p - \langle z,x\rangle_p\right| \le \|y\|\,\|x\| \tag{1.11}$$

for each $x,y,z\in X$ and $p=s$ or $p=i$.

One of the most used inequalities in normed spaces is the triangle inequality for several vectors, i.e.,

$$\left\|\sum_{j=1}^n x_j\right\| \le \sum_{j=1}^n \|x_j\| \tag{1.12}$$

for any $x_j\in X$, $j\in\{1,\dots,n\}$.

The main aim of this paper is to point out some inequalities for norms of the vectors x_j and $\sum_{j=1}^{n} x_j$, including some reverses of the triangle inequality in the multiplicative form, i.e., lower bounds for the quantity

$$\frac{\left\|\sum_{j=1}^{n} x_j\right\|}{\sum_{j=1}^{n} \|x_j\|},$$

provided that not all x_j are zero and satisfy some appropriate conditions.

For classical results related to the reverse of the triangle inequality in normed spaces see [3], [7], [9] and [8]. For more recent results, see [5], [6], [1] and [2].

2. The results

The following lemma is of interest itself as well.

Lemma 2.1. *Let $(X, \|\cdot\|)$ be a normed linear space. If $x, a \in X$, then*

$$\langle x, a\rangle_i \geq \frac{1}{2}\left(\|a\|^2 - \|x-a\|^2\right). \tag{2.1}$$

If $\|a\| > \|x-a\|$, then the constant $\frac{1}{2}$ cannot be replaced by a larger quantity.

Proof. Utilising the semi-inner product properties, we have by (1.7) that

$$\langle x, a\rangle_i = \lim_{s\to 0-} \frac{\|a+sx\|^2 - \|a\|^2}{2s} \geq \frac{\|a+(-1)\,x\|^2 - \|a\|^2}{2\,(-1)} = \frac{\|a\|^2 - \|x-a\|^2}{2}$$

and the inequality (2.1) is proved.

Now, assume that $\|a\| > \|x-a\|$ and there exists a $C > 0$ with the property that

$$\langle x, a\rangle_i \geq C\left(\|a\|^2 - \|x-a\|^2\right). \tag{2.2}$$

Obviously $a \neq 0$, and if we choose $x = \varepsilon a$, $\varepsilon \in (0,1)$, then $\|a\| > \|x-a\|$ since $\|x-a\| = (1-\varepsilon)\|a\|$. Replacing x by εa in (2.2) we get

$$\varepsilon\|a\|^2 \geq C\left(\|a\|^2 - (1-\varepsilon)^2\|a\|^2\right)$$

giving

$$\varepsilon \geq C\left(2\varepsilon - \varepsilon^2\right),$$

for any $\varepsilon \in (0,1)$. This is in fact $1 \geq C(2-\varepsilon)$ and if we let $\varepsilon \to 0+$, we get $C \leq \frac{1}{2}$. $\square$

Remark 2.2. As a coarser, but maybe more useful inequality, we can state that

$$\langle x, a\rangle_i \geq \frac{1}{2}\|x\|\left(\|a\| - \|x-a\|\right), \tag{2.3}$$

provided $\|a\| \geq \|x-a\|$.

We observe that (2.3) follows from (2.1) since, for $\|a\| \geq \|x-a\|$, the triangle inequality gives:

$$\begin{aligned}\frac{1}{2}\left(\|a\|^2-\|x-a\|^2\right) &= \frac{1}{2}\left(\|a\|-\|x-a\|\right)\left(\|a\|+\|x-a\|\right)\\ &\geq \frac{1}{2}\left(\|a\|-\|x-a\|\right)\|x\|.\end{aligned}$$

It is an open question whether the constant $\frac{1}{2}$ in (2.3) is sharp.

The following result may be stated.

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a normed space and $x_j \in X$, $j \in \{1, \dots, n\}$, $a \in X \setminus \{0\}$. Then for any $p_j \geq 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^n p_j = 1$ we have*

$$\left\|\sum_{j=1}^n p_j x_j\right\|\|a\| + \frac{1}{2}\sum_{j=1}^n p_j \|x_j - a\|^2 \geq \frac{1}{2}\|a\|^2. \tag{2.4}$$

The constant $\frac{1}{2}$ in the right hand side of (2.4) is best possible in the sense that it cannot be replaced by a larger quantity.

Proof. We apply Lemma 2.1 on stating that

$$\langle x_j, a\rangle_i + \frac{1}{2}\|x_j - a\|^2 \geq \frac{1}{2}\|a\|^2$$

for each $j \in \{1, \dots, n\}$.

Multiplying with $p_j \geq 0$ and summing over j from 1 to n, we get

$$\sum_{j=1}^n p_j \langle x_j, a\rangle_i + \frac{1}{2}\sum_{j=1}^n p_j \|x_j - a\|^2 \geq \frac{1}{2}\|a\|^2 \sum_{j=1}^n p_j. \tag{2.5}$$

Utilising the superadditivity property of the semi-inner product $\langle \cdot, \cdot\rangle_i$ in the first variable (see [4, p. 29]) we have

$$\left\langle \sum_{j=1}^n p_j x_j, a\right\rangle_i \geq \sum_{j=1}^n p_j \langle x_j, a\rangle_i. \tag{2.6}$$

By the Schwarz inequality applied for $\sum_{j=1}^n p_j x_j$ and a, we also have

$$\left\|\sum_{j=1}^n p_j x_j\right\|\|a\| \geq \left\langle \sum_{j=1}^n p_j x_j, a\right\rangle_i. \tag{2.7}$$

Therefore, by (2.5)–(2.7) we deduce the desired inequality (2.4).

Now assume that there exists a $D > 0$ with the property that

$$\left\| \sum_{j=1}^{n} p_j x_j \right\| \|a\| + \frac{1}{2} \sum_{j=1}^{n} p_j \|x_j - a\|^2 \geq D \|a\|^2 , \tag{2.8}$$

for any $n \geq 1$, $x_j \in X$, $p_j \geq 0$, $j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ and $a \in X \setminus \{0\}$.

If in (2.8) we choose $n = 1$, $p_1 = 1$, $x_1 = \varepsilon a$, $\varepsilon \in (0, 1)$, then we get

$$\varepsilon \|a\|^2 + \frac{1}{2} (1 - \varepsilon)^2 \|a\|^2 \geq D \|a\|^2 ,$$

giving

$$\varepsilon + \frac{1}{2} (1 - \varepsilon)^2 \geq D,$$

for any $\varepsilon \in (0, 1)$. Letting $\varepsilon \to 0+$, we deduce $D \leq \frac{1}{2}$ and the proof is complete. □

The following result may be stated as well:

Proposition 2.4. *Let $x_j, a \in X$ with $a \neq 0$ and $\|x_j - a\| \leq \|a\|$ for each $j \in \{1, \ldots, n\}$. Then for any $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ we have*

$$\left\| \sum_{j=1}^{n} p_j x_j \right\| \|a\| + \frac{1}{2} \sum_{j=1}^{n} p_j \|x_j\| \|x_j - a\| \geq \frac{1}{2} \|a\| \sum_{j=1}^{n} p_j \|x_j\| . \tag{2.9}$$

Proof. From (2.3) we have

$$\langle x_j, a \rangle_i + \frac{1}{2} \|x_j\| \|x_j - a\| \geq \frac{1}{2} \|a\| \|x_j\|$$

for any $j \in \{1, \ldots, n\}$.

The proof follows in the same manner as in Theorem 2.3 and we omit the details. □

The following reverse of the generalised triangle inequality may be stated:

Theorem 2.5. *Let $x_j \in X \setminus \{0\}$ and $a \in X \setminus \{0\}$ such that $\|a\| \geq \|x_j - a\|$ for each $j \in \{1, \ldots, n\}$. Then for any $p_j \geq 0, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ we have*

$$\frac{\left\| \sum_{j=1}^{n} p_j x_j \right\|}{\sum_{j=1}^{n} p_j \|x_j\|} \geq \frac{1}{2} \min_{1 \leq j \leq n} \left\{ \frac{\|a\|^2 - \|a - x_j\|^2}{\|x_j\| \|a\|} \right\} (\geq 0) . \tag{2.10}$$

The constant $\frac{1}{2}$ is best possible in (2.10).

Proof. Let us denote

$$\rho := \min_{1\le j\le n}\left\{\frac{\|a\|^2-\|a-x_j\|^2}{\|x_j\|}\right\}.$$

From Lemma 2.1 we have

$$\frac{\langle x_j,a\rangle_i}{\|x_j\|}\ge\frac{1}{2}\cdot\frac{\|a\|^2-\|x_j-a\|^2}{\|x_j\|}\ge\frac{1}{2}\rho$$

for each $j\in\{1,\dots,n\}$. Therefore

$$\langle x_j,a\rangle_i\ge\frac{1}{2}\rho\|x_j\|,\quad j\in\{1,\dots,n\}.$$

Multiplying with p_j and summing over j from 1 to n we obtain

$$\sum_{j=1}^n p_j\langle x_j,a\rangle_i\ge\frac{1}{2}\rho\sum_{j=1}^n p_j\|x_j\|, \tag{2.11}$$

and since:

$$\left\|\sum_{j=1}^n p_jx_j\right\|\|a\|\ge\left\langle\sum_{j=1}^n p_jx_j,a\right\rangle_i\ge\sum_{j=1}^n p_j\langle x_j,a\rangle_i, \tag{2.12}$$

hence by (2.11) and (2.12) we deduce the desired result (2.10).

Now, assume that there exists a constant $E>0$ such that

$$\frac{\left\|\sum_{j=1}^n p_jx_j\right\|}{\sum_{j=1}^n p_j\|x_j\|}\ge E\cdot\min_{1\le j\le n}\left\{\frac{\|a\|^2-\|a-x_j\|^2}{\|x_j\|\|a\|}\right\}, \tag{2.13}$$

provided $\|a\|\ge\|x_j-a\|$, $j\in\{1,\dots,n\}$.

If we choose $x_1=\dots=x_n=\varepsilon a$, $\varepsilon\in(0,1)$, and $p_1=...=p_n=\frac{1}{n}$, then we get

$$1\ge E\cdot\frac{\|a\|^2-(1-\varepsilon)^2\|a\|^2}{\varepsilon\|a\|^2},$$

giving

$$1\ge E(2-\varepsilon)$$

for any $\varepsilon\in(0,1)$. Letting $\varepsilon\to0+$, we deduce $E\le\frac{1}{2}$ and the proof is complete. □

The following result may be stated as well:

Proposition 2.6. *Let $x_j, a \in X \setminus \{0\}$, $j \in \{1, \dots, n\}$ such that $\|x_j - a\| \leq \|a\|$. Then for any $p_j \geq 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ we have*

$$\frac{\left\|\sum_{j=1}^{n} p_j x_j\right\|}{\sum_{j=1}^{n} p_j \|x_j\|} \geq \frac{\left(\|a\| - \max_{1\leq j\leq n} \|x_j - a\|\right)}{2\|a\|} \quad (\geq 0). \tag{2.14}$$

Proof. From (2.3) we have

$$\begin{aligned} \frac{\langle x_j, a\rangle_i}{\|x_j\|} &\geq \frac{1}{2}\left(\|a\| - \|x_j - a\|\right) \\ &\geq \frac{1}{2}\min_{1\leq j\leq n}\left(\|a\| - \|x_j - a\|\right) \\ &= \frac{1}{2}\left(\|a\| - \max_{1\leq j\leq n}\|x_j - a\|\right). \end{aligned}$$

Now the proof follows the same steps as in that of Theorem 2.3 and the details are omitted. □

Remark 2.7. If $\|a\| = 1$ and $\|x_j - a\| \leq 1$, then (2.10) has a simpler form:

$$\frac{\left\|\sum_{j=1}^{n} p_j x_j\right\|}{\sum_{j=1}^{n} p_j \|x_j\|} \geq \frac{1}{2}\min_{1\leq j\leq n}\left\{\frac{1 - \|x_j - a\|^2}{\|x_j\|}\right\} (\geq 0), \tag{2.15}$$

while (2.14) becomes

$$\frac{\left\|\sum_{j=1}^{n} p_j x_j\right\|}{\sum_{j=1}^{n} p_j \|x_j\|} \geq \frac{1}{2}\left(1 - \max_{1\leq j\leq n}\|x_j - a\|\right) (\geq 0). \tag{2.16}$$

A different approach for bounding the semi-inner product is incorporated in the following:

Lemma 2.8. *Let $(X, \|\cdot\|)$ be a normed space. If $x, a \in X$, then*

$$\langle x, a\rangle_i \geq \|a\|\left(\|a\| - \|x - a\|\right). \tag{2.17}$$

The inequality (2.17) is sharp.

Proof. If $a = 0$, then obviously (2.17) holds with equality. For $a \neq 0$, consider

$$\tau_-(x, a) := \lim_{s\to 0-}\frac{\|a + sx\| - \|a\|}{s}.$$

Observe that

$$\begin{aligned}\langle x, a\rangle_i &= \lim_{s\to 0-} \frac{\|a+sx\|^2 - \|a\|^2}{2s} \\ &= \tau_-(x,a) \lim_{s\to 0-} \left[\frac{\|a+sx\| + \|a\|}{2}\right] \\ &= \tau_-(x,a)\|a\|.\end{aligned} \tag{2.18}$$

On the other hand, since the function $R \ni s \longmapsto \|a+sx\| \in \mathbb{R}_+$ is convex on $\mathbb{R}$, hence

$$\tau_-(x,a) \geq \frac{\|a+(-1)x\| - \|a\|}{(-1)} = \|a\| - \|x-a\|. \tag{2.19}$$

Consequently, by (2.18) and (2.19) we get (2.17).

Now, let $x = \varepsilon a$, $\varepsilon \in (0,1)$, $a \neq 0$. Then

$$\langle x, a\rangle_i = \varepsilon \|a\|^2, \quad \|a\| - \|x-a\| = \|a\| - (1-\varepsilon)\|a\| = \varepsilon\|a\|,$$

which shows that the equality case in (2.17) holds true for the nonzero quantities $\varepsilon\|a\|^2$. The proof is complete. □

The following reverse of the generalised triangle inequality may be stated.

Theorem 2.9. *Let $a, x_j \in X\setminus\{0\}$ for $j \in \{1,\dots,n\}$ with the property that $\|a\| \geq \|x_j - a\|$ for $j \in \{1,\dots,n\}$. Then for any $p_j \geq 0, j \in \{1,\dots,n\}$ with $\sum_{j=1}^n p_j = 1$ we have*

$$\frac{\left\|\sum_{j=1}^n p_j x_j\right\|}{\sum_{j=1}^n p_j \|x_j\|} \geq \min_{1\leq j\leq n}\left\{\frac{\|a\| - \|x_j - a\|}{\|x_j\|}\right\} \quad (\geq 0). \tag{2.20}$$

The inequality (2.20) is sharp.

Proof. On making use of Lemma 2.8, we have:

$$\begin{aligned}\frac{\langle x_j, a\rangle_i}{\|x_j\|} &\geq \|a\|\left(\frac{\|a\| - \|x_j - a\|}{\|x_j\|}\right) \\ &\geq \|a\|\eta,\end{aligned}$$

for each $j \in \{1,\dots,n\}$, where

$$\eta := \min_{1\leq j\leq n}\left\{\frac{\|a\| - \|x_j - a\|}{\|x_j\|}\right\}.$$

Now utilising the same argument explained in the proof of Theorem 2.5, we get the desired inequality (2.20).

If we choose in (2.20) $x_1 = \cdots = x_n = \varepsilon a$, $\varepsilon \in (0,1)$, $a \neq 0$, and $p_1 = ... = p_n = 1$ then we have equality, and the proof is complete. □

Remark 2.10. The above result may be stated in a simpler way, i.e., if $\rho \in (0,1)$, a and $x_j \in X \setminus \{0\}$, $j \in \{1,\ldots,n\}$ are such that

$$(\|x_j\| \geq)\, \|a\| - \|x_j - a\| \geq \rho \|x_j\| \quad (\geq 0) \tag{2.21}$$

for each $j \in \{1,\ldots,n\}$, then

$$\left\| \sum_{j=1}^{n} p_j x_j \right\| \geq \rho \sum_{j=1}^{n} p_j \|x_j\|. \tag{2.22}$$

3. Other related results to the triangle inequality

The following result may be stated:

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a normed linear space and $x_1,\ldots,x_n$ nonzero vectors in X and $p_j \geq 0$ with $\sum_{j=1}^{n} p_j = 1$. If $\bar{x}_p := \sum_{j=1}^{n} p_j x_j \neq 0$ and there exists a number $r > 0$ with*

$$\frac{\langle x_j, \bar{x}_p \rangle_i}{\|x_j\| \|\bar{x}_p\|} \geq r \quad \text{for each} \;\; j \in \{1,\ldots,n\}, \tag{3.1}$$

then

$$\left\| \sum_{j=1}^{n} p_j x_j \right\| \geq r \sum_{j=1}^{n} p_j \|x_j\|. \tag{3.2}$$

If $p_j > 0$ for each $j \in \{1,\ldots,n\}$, then the equality holds in (3.2) if and only if the equality case hold in (3.1) for each $j \in \{1,\ldots,n\}$.

Proof. From (3.1) on multiplying with $p_i \geq 0$ we have

$$\langle p_j x_j, \bar{x}_p \rangle_i \geq r p_j \|\bar{x}_p\| \|x_j\|$$

for any $j \in \{1,\ldots,n\}$.

Summing over j from 1 to n and taking into account the superadditivity property of the interior semi-inner product, we have

$$\left\langle \sum_{j=1}^{n} p_j x_j, \bar{x}_p \right\rangle_i \geq \sum_{j=1}^{n} \langle p_j x_j, \bar{x}_p \rangle_i \geq r \|\bar{x}_p\| \sum_{j=1}^{n} p_j \|x_j\| \tag{3.3}$$

and since

$$\left\langle \sum_{j=1}^{n} p_j x_j, \bar{x}_p \right\rangle_i = \left\| \sum_{j=1}^{n} p_j x_j \right\|^2 \neq 0$$

hence by (3.3) we get (3.2).

The equality case is obvious and the proof is complete. $\square$

For the system of vectors $x_1, \ldots, x_k \in X$, we denote by $\bar{x}$ its gravity center, i.e.,

$$\bar{x} := \frac{1}{n} \sum_{i=1}^{n} x_i.$$

The following corollary is obvious.

Corollary 3.2. *Let $x_1, \ldots, x_n \in X \backslash \{0\}$ be such that $\bar{x} \neq 0$. If there exists a number $r > 0$ such that*

$$\frac{\langle x_j, \bar{x} \rangle_i}{\|x_j\| \|\bar{x}\|} \geq r \quad \text{for each} \;\; j \in \{1, \ldots, n\}, \tag{3.4}$$

then the following reverse of the generalised triangle inequality holds:

$$\left\| \sum_{j=1}^{n} x_j \right\| \geq r \sum_{j=1}^{n} \|x_j\|. \tag{3.5}$$

The equality holds in (3.5) if and only if the case of equality holds in (3.4) for each $j \in \{1, \ldots, n\}$.

The following refinements of the generalised triangle inequality may be stated as well:

Theorem 3.3. *Let $x_i, \bar{x}_p, p_i,\ i \in \{1, \ldots, n\}$ be as in Theorem 3.1. If there exists a number R with $1 > R > 0$ and such that*

$$R \geq \frac{\langle x_j, \bar{x}_p \rangle_s}{\|x_j\| \|\bar{x}_p\|} \quad \text{for each} \;\; j \in \{1, \ldots, n\}, \tag{3.6}$$

then

$$R \sum_{j=1}^{n} p_j \|x_j\| \geq \left\| \sum_{j=1}^{n} p_j x_j \right\|. \tag{3.7}$$

If $p_j > 0$ for each $j \in \{1, \ldots, n\}$, then the equality holds in (3.7) if and only if the equality case holds in (3.6) for each $j \in \{1, \ldots, n\}$.

The proof is similar to the one in Theorem 3.1 on taking into account that the superior semi-inner product is a subadditative functional in the first variable.

Corollary 3.4. *Let $x_j,\ j \in \{1, \ldots, n\}$ be as in Corollary 3.2. If there exists an R with $1 > R > 0$ and*

$$R \geq \frac{\langle x_j, \bar{x} \rangle_s}{\|x_j\| \|\bar{x}\|} \quad \text{for each} \;\; j \in \{1, \ldots, n\}, \tag{3.8}$$

then the following refinement of the generalised triangle inequality holds:

$$R \sum_{j=1}^{n} \|x_j\| \geq \left\| \sum_{j=1}^{n} x_j \right\|. \tag{3.9}$$

The equality hold in (3.9) if and only if the case of equality holds in (3.8) for each $j \in \{1, \dots, n\}$.

References

[1] A. H. Ansari and M. S. Moslehian, *Refinements of the triangle inequality in Hilbert and Banach spaces*, J. Inequal. Pure & Appl. Math., **6**(2005), No. 3, Article 64. [Online http://jipam.vu.edu.au/article.php?sid=537].

[2] A. H. Ansari and M. S. Moslehian, *More on reverse triangle inequality in inner product spaces*, Int. J. Math. Math. Sci., **18**(2005), 2883-2893.

[3] J. B. Diaz and F. T. Metcalf, *A complementary triangle inequality in Hilbert and Banach spaces*, Proc. Amer. Math. Soc., **17**(1) (1966), 88-97.

[4] S. S. Dragomir, *Semi-Inner Products and Applications*, Nova Science Publishers, Inc., New York, 2004.

[5] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces,* Nova Science Publishers, Inc., New York, 2004.

[6] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces*, RGMIA Monographs, Victoria University, 2005. (ONLINE: http://rgmia.vu.edu.au/monographs/).

[7] S. M. Khaleelula, *On Diaz-Metcalf's complementary triangle inequality*, Kyungpook Math. J., **15** (1975), 9-11.

[8] P. M. Miličić, *On a complementary inequality of the triangle inequality (French)*, Mat. Vesnik, **41**(2) (1984), 83-88.

[9] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.