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CRITICAL POINT THEORY ON QUASI-METRIC SPACES

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Abstract. In this paper we extend the notions of weak slope and critical point for quasi-metric spaces. We study properties of critical set. We establish deformation and stability results.

1. Preliminaries

Let X be a non-empty set. Recall that a quasi-metric on X is a non-negative function ρ on $X \times X$ such that

(i) for every $x, y \in X,\ \rho(x, y) = 0$ iff $x = y$

(ii) for every $x, y \in X,\ \rho(x, y) = \rho(y, x)$

(iii) there exists a finite constant $C \geq 1$ such that for every $x, y, z \in X$

$$\rho(x, z) \leq C[\rho(x, y) + \rho(y, z)].$$

We call (X, ρ) a quasi-metric space. Let us remark that (iii) is equivalent with

$(iii)'$ there exists a finite constant $\widetilde{C}$ such that for every $x, y, z \in X$

$$\rho(x, z) \leq \widetilde{C} \max\{\rho(x, y), \rho(y, z)\}.$$

For example, if $\|x - y\|$ denotes the usual euclidean distance between $x, y \in \mathbb{R}^n$, then $\rho(x, y) = \|x - y\|^n$ satisfies $\rho(x, z) \leq 2^n \max\{\rho(x, y), \rho(y, z)\} \leq 2^n[\rho(x, y) + \rho(y, z)],\ \forall\ x, y, z \in\ \mathbb{R}^n$, so it is a quasi-metric on $\mathbb{R}^n$.

It is known that quasi-metric spaces are metrizable. The proof was given by Gustavsson in [13]. An important result concerning quasi-metrics was given by Macias and Segovia in [14], which proved that any quasi-metric on X produces

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an uniform structure on $X \times X$ with a countable basis generating a metrizable topology on X and were able to introduce non-trivial Lipschitz functions on quasi-metric spaces. A more explicit construction of the metric induced by a quasi-metric was given by Aimar, Iaffei and Nitti in [1] without the use of uniformities. They also introduced a generalization of quasi-metric spaces and proved that any generalized quasi-metric space is metrizable.

Let (X, ρ) be a quasi-metric space. Denote by $B_r(x)$ the ball with center in x and radius r : $B_r(x) = \{y \in X|\ \rho(x, y) < r\}$. A subset A of X is open if $\forall\ x \in A,\ \exists\ \varepsilon > 0$ such that $B_\varepsilon(x) \subset A$. Remark that for $C > 1$ it does not follows that $B_r(x)$ is open. Following [14], there exists a quasi-metric ρ' on X such that ρ and ρ' are equivalent in the sense that there exist two positive constants c_1, c_2 satisfying $c_1\rho(x, y) \leq \rho'(x, y) \leq c_2\rho(x, y),\ \forall\ x, y \in X$ and the balls corresponding to ρ' are open sets in the topology induced by ρ'. Then, without loss of generality, we can always assume that the balls $B_r(x)$ are open.

We mention that if ρ is a quasi-metric on X, it is standard to verify that $\widetilde{\rho} = \dfrac{\rho}{1+\rho}$ is also a quasi-metric on X. Then we can always replace ρ by a bounded quasi-metric.

2. ρ-weak slope and ρ-strong slope

In this section we extend some definitions and results developed by Corvellec, Degiovanni and Marzocchi in [6]-[12].

Through (X, ρ) denotes a quasi-metric space.

Definition 2.1. Let $f : X \to \mathbb{R}$ be continuous and $x \in X$. Define

$$|df|(x) = \sup\{\sigma \geq 0 \mid \exists\, \delta > 0,\ \exists\, \mathcal{H} : B_\delta(x) \times [0, \delta] \to X \text{ continuous, such that } \rho(\mathcal{H}(y, t), y) \leq t,\ f(\mathcal{H}(y, t)) \leq f(y) - \sigma t,\ \forall\ y \in B_\delta(x),\ \forall\ t \in [0, \delta]\}.$$

We call $|df|(x)$ the ρ-weak slope of f at x.

Let us consider

$$\overline{\rho}((x, t), (x', t')) = \sqrt{\rho(x, x')^2 + (t - t')^2}.$$

For $(x, t), (x', t'), (x'', t'') \in X \times \mathbb{R}$ arbitrary, we can write

$$\begin{aligned}
\overline{\rho}((x, t), (x'', t'')) &= \sqrt{\rho(x, x'')^2 + (t - t'')^2} \\
&\leq \sqrt{C^2(\rho(x, x') + \rho(x', x''))^2 + ((t - t') + (t' - t''))^2} \\
&\leq \sqrt{C^2\rho(x, x')^2 + (t - t')^2} + \sqrt{C^2\rho(x', x'')^2 + (t' - t'')^2} \\
&\leq C[\overline{\rho}((x, t), (x', t')) + \overline{\rho}((x', t'), (x'', t''))].
\end{aligned}$$

We conclude that $\overline{\rho}$ is a quasi-metric on $X \times \mathbb{R}$. Denote

$$epi(f) = \{(x, \xi) \in X \times \mathbb{R} | \ f(x) \leq \xi\}$$

and define

$$\mathcal{G}_f(x, \xi) = \xi.$$

Then $\overline{\rho}$ is a quasi-metric on $epi(f)$.

Let $f : X \to \mathbb{R}$ be continuous. Then we follow step by step the proof of [12, Proposition 2.3] and we obtain:

$$|d\mathcal{G}_f|(x, f(x)) = \begin{cases} \dfrac{|df|(x)}{\sqrt{1 + (|df|(x))^2}}, & |df|(x) < +\infty \\ 1, & |df|(x) = +\infty. \end{cases}$$

This result allows to define the ρ-weak slope for a lower ρ-semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ by taking

$$|df|(x) = \begin{cases} \dfrac{|d\mathcal{G}_f|(x, f(x))}{\sqrt{1 - (|d\mathcal{G}_f|(x, f(x)))^2}}, & |d\mathcal{G}_f|(x, f(x)) < 1 \\ +\infty, & |d\mathcal{G}_f|(x, f(x)) = 1. \end{cases}$$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. If $(x_n), x_n \in X$ is a sequence converging to $x \in X$ such that $(f(x_n))$ converges to $f(x)$, then $|df|(x) \leq \liminf_n |df|(x_n)$; we conclude that the ρ-weak slope is lower semicontinuous.

Definition 2.2. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. Define the ρ-strong slope of f at $x \in X$ by taking

$$|\nabla f|(x) = \begin{cases} 0, & \text{if } x \text{ is a local minimum of } f \\ \limsup\limits_{y \to x} \dfrac{f(y) - f(x)}{\rho(y, x)}, & \text{otherwise.} \end{cases}$$

Then $|df|(x) \leq |\nabla f|(x), \ \forall \ x \in X$. Moreover, for a C^1-Finsler manifold and $f \in C^1(X, \mathbb{R})$ it follows that $|df|(x) = |\nabla f|(x) = \|(df)_x\|, \ \forall \ x \in X$. The proof is analog to [12, Theorem 2.11] and [12, Corollary 2.12].

3. ρ-critical points

Critical point theory on metric spaces was developed by Corvellec, Degiovanni and Marzocchi in [6]-[12].

Let (X, ρ) be a quasi-metric space and let $f : X \to \mathbb{R}$ be continuous.

Definition 3.1. A point $x \in X$ is called ρ-critical point of f if $|df|(x) = 0$. A real number c is a ρ-critical value of f if there exists $x \in X$ such that $|df|(x) = 0$ and $f(x) = c$.

An useful tool in critical point theory is the Palais-Smale condition.

Definition 3.2. We say that f satisfies the Palais-Smale condition at level c ($c \in \mathbb{R}$ fixed), denoted by $(PS)_c$, if every sequence (x_n) in X such that $|df|(x_n) \to 0$ and $f(x_n) \to c$ has a subsequence (x_{n_k}) convergent in X (to a ρ-critical point of f).

Denote by $K[f]$ the ρ-critical set of f. Define the ρ-critical set of level c by $K_c[f] = K[f] \cap f^{-1}(c)$; the set of sublevel c of f is $X_c[f] = \{x \in X |\ f(x) \leq c\}$.

Theorem 3.3. *The ρ-critical set $K[f]$ is closed in X.*

Proof. Let $x_0 \in X \setminus K[f]$ be fixed. Then $|df|(x_0) \neq 0$. Let $\sigma > 0$ such that $|df|(x_0) > \sigma$. Because $|df|$ is a lower semicontinuous function we can take U_0 an open neighborhood of x_0 in X such that $|df|(x) > \sigma,\ \forall\ x \in U_0$. Then $|df|(x) \neq 0,\ \forall\ x \in U_0$. □

Remark 3.4. Let (X, ρ) be a quasi-metric space and $f : X \to \mathbb{R}$ be a continuous function. If x_0 is a local minimum point of f, then x_0 is a ρ-critical point of f.

4. An embedding theorem

Theorem 4.1. *For any quasi-metric space (X, ρ), there exists a Banach space E and $h : X \to E$ such that for any $x_1, x_2 \in X$ we have*

$$\|h(x_1) - \frac{1}{C}h(x_2)\| = \rho(x_1, x_2) = \|\frac{1}{C}h(x_1) - h(x_2)\|.$$

Proof. We can assume that ρ is a bounded quasi-metric. Consider the set

$$E = \{f : X \to \mathbb{R} |\ f \text{ bounded }\}$$

endowed with the norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Let $x \in X$ be arbitrary. Define

$$f_x : X \to \mathbb{R},\ y \mapsto f_x(y) = \rho(x, y)$$

$$h : X \to E, x \mapsto h(x) = f_x.$$

We deduce that $\|h(x_1) - Ch(x_2)\| = \sup_{y \in X} |\rho(x_1, y) - C\rho(x_2, y)| \leq C\rho(x_1, x_2)$, $\forall x_1, x_2 \in X$. For $y = x_1$ it results that $\|h(x_1) - Ch(x_2)\| \geq C\rho(x_1, x_2)$.

We conclude that $\|h(x_1) - Ch(x_2)\| = C\rho(x_1, x_2)$.

The inequality $\|Ch(x_1) - h(x_2)\| \geq C\rho(x_1, x_2)$ can be proved analogous. □

5. Change of quasi-metric

It is interesting to know the relationship between weak slopes of a given function in a fixed point with respect to equivalent quasi-metrics.

Theorem 5.1. *Let ρ and ρ' quasi-metrics on X such that there exist positive constants c_1, c_2 satisfying $c_1\rho(x,y) \leq \rho'(x,y) \leq c_2\rho(x,y)$, $\forall\ x, y \in X$. If $f : X \to \mathbb{R}$ is a continuous function, then*

$$\frac{1}{c_2}|df|(x) \leq |\widetilde{df}|(x) \leq \frac{1}{c_1}|df|(x)$$

at any point $x \in X$.

Proof. Consider $\sigma > 0$ such that $|df|(x) > \sigma$. By definition, there exist $\delta > 0$ and $\mathcal{H} : B_\delta(x) \times [0,\delta] \to X$ continuous such that $\rho(\mathcal{H}(y,t), y) \leq t$ and $f(\mathcal{H}(y,t)) \leq f(y) - \sigma t$, $\forall\ y \in B_\delta(x)$, $\forall\ t \in [0,\delta]$. Take $\widetilde{\delta}$ such that $0 < \widetilde{\delta} < c_2\delta$ and define $\widetilde{\mathcal{H}} : \widetilde{B}_{\widetilde{\delta}}(x) \times [0,\widetilde{\delta}] \to X$, $\widetilde{\mathcal{H}}(y,t) = \mathcal{H}(y, \frac{t}{c_2})$. Then we obtain the following inequalities:

$$\rho'(\widetilde{\mathcal{H}}(y,t), y) = \rho'(\mathcal{H}(y, \frac{t}{c_2}), y) \leq c_2 \cdot \rho(\mathcal{H}(y, \frac{t}{c_2}), y) \leq c_2 \cdot \frac{t}{c_2} = t$$

$$f(\widetilde{\mathcal{H}}(y,t)) = f(\mathcal{H}(y, \frac{t}{c_2})) \leq f(y) - \sigma \cdot \frac{t}{c_2}.$$

We conclude that $|\widetilde{df}|(x) \geq \dfrac{\sigma}{c_2}$ for any σ and, consequently, it follows that

$$|\widetilde{df}|(x) \geq \frac{1}{c_2}|df|(x).$$

Take now $\sigma > 0$ such that $|\widetilde{df}|(x) > \sigma$. Then there exist $\widetilde{\delta} > 0$ and a continuous function $\widetilde{\mathcal{H}} : \widetilde{B}_{\widetilde{\delta}}(x) \times [0,\widetilde{\delta}] \to X$ such that $\rho(\widetilde{\mathcal{H}}(y,t), y) \leq t$ and$f(\widetilde{\mathcal{H}}(y,t)) \leq f(y) - \sigma t$, $\forall\ y \in \widetilde{B}_{\widetilde{\delta}}(x)$, $\forall\ t \in [0,\widetilde{\delta}]$. Let δ be a positive number satisfying $c_1\delta < \widetilde{\delta}$. Define $\mathcal{H} : B_\delta(x) \times [0,\delta] \to X$ by taking $\mathcal{H}(y,t) = \widetilde{\mathcal{H}}(y, c_1 t)$. Then the following inequalities are verified:

$$\rho(\mathcal{H}(y,t), y) = \rho(\widetilde{\mathcal{H}}(y, c_1 t), y) \leq \frac{1}{c_1} \cdot \rho'(\widetilde{\mathcal{H}}(y, c_1 t), y) \leq \frac{1}{c_1} \cdot c_1 t = t$$

$$f(\mathcal{H}(y,t)) = f(\widetilde{\mathcal{H}}(y, c_1 t)) \leq f(y) - \sigma \cdot c_1 t.$$

These inequalities implies $|df|(x) \geq c_1\sigma$; σ being arbitrary, it follows that $|df|(x) \geq c_1|\widetilde{df}|(x)$. □

We prove now the change of quasi-metric principle, following the metric case given by Corvellec in [6]-[8].

Let (X, ρ) be a quasi-metric space.

Theorem 5.2. *Let A be a nonempty subset of X and $\beta : [0,\infty) \to (0,\infty)$ continuous.*

There exists a quasi-metric $\widetilde{\rho}$ on X which is equivalent to ρ on X and satisfy the following properties:

(i) if $f : X \to \mathbb{R}$ is continuous, then

$$|\widetilde{df}|(x) = \frac{|df|(x)}{\beta(\rho(x,A))}, \ \forall\, x \in X.$$

(ii) if $f : X \to \mathbb{R}$ is lower semicontinuous, then

$$|\widetilde{\nabla f}|(x) = \frac{|\nabla f|(x)}{\beta(\rho(x,A))}, \ \forall\, x \in X.$$

Proof. The construction of $\widetilde{\rho}$ is standard: for $x, y \in X$ denote by Γ_{xy} the set of all C^1-paths $[0,1] \to X$ from x to y and define

$$\widetilde{\rho}(x,y) = \inf_{\alpha \in \Gamma_{xy}} \int_0^1 \beta(\rho(\alpha(t),A))\|\alpha'(t)\|dt,$$

where $\rho(x_0, A) = \inf\limits_{a \in A} \rho(x_0, a)$. See [8, Theorem 3]. This is a quasi-metric on X, equivalent to ρ on X.

(i) Let x be fixed in X and let ε be a positive number. Assume that $|df|(x) > \sigma > 0$. Then there exist $\delta > 0$ and $\mathcal{H} : B_\delta(x) \times [0,\delta] \to X$ continuous such that $\|\mathcal{H}(y,t) - \frac{1}{C}y\| \le t,\ f(\mathcal{H}(y,t)) \le f(y) - \sigma t$.

Take $\delta > 0$ such that $\beta(\rho(y,A)) \le \beta(\rho(x,A)) + \varepsilon$ for $y \in B_{2\delta}(x)$ and take $\widetilde{\delta} > 0$ such that $\widetilde{B}_{\widetilde{\delta}}(x) \subset B_\delta(x)$ and $\widetilde{\delta} < \delta(\beta(\rho(x,A)) + \varepsilon)$.

Define $\widetilde{\mathcal{H}} : \widetilde{B}_{\widetilde{\delta}}(x) \times [0,\widetilde{\delta}] \to X$ by

$$\widetilde{\mathcal{H}}(y,t) = \mathcal{H}\left(y, \frac{t}{\beta(\rho(x,A)) + \varepsilon}\right).$$

For $(y,t) \in \widetilde{B}_{\widetilde{\delta}}(x)$ and $s \in [0,1]$ consider the path $\alpha(s) = y + s(\widetilde{H}(y,t) - \frac{1}{C}y)$. Then

$$\widetilde{\rho}(\widetilde{H}(y,t),y) \le \int_0^1 \beta(\rho(\alpha(s),A))\|\alpha'(s)\|ds \le t$$

and

$$f(\widetilde{H}(y,t)) \le f(y) - \frac{\sigma t}{\beta(\rho(x,A)) + \varepsilon},$$

which shows that

$$|\widetilde{df}|(x) \ge \frac{\sigma}{\beta(\rho(x,A)) + \varepsilon}.$$

We conclude that

$$|\widetilde{df}|(x) \ge \frac{|df|(x)}{\beta(\rho(x,A))}.$$

Conversely, let x be fixed in X and let ε be a positive number such that $\beta(\rho(x,A)) > \varepsilon$. Take $\delta > 0$ such that $\beta(\rho(x,A)) - \varepsilon \leq \beta(\rho(y,A))$ for $y \in B_{3\delta}(x)$ and take $\sigma > 0$ such that $|\widetilde{df}|(x) > \sigma > 0$.

Let $\widetilde{\delta} > 0$ and $\widetilde{\mathcal{H}} : \widetilde{B}_{\widetilde{\delta}}(x) \times [0, \widetilde{\delta}] \to X$ be continuous such that $\widetilde{\rho}(\widetilde{H}(y,t), y) \leq t$ and $f(\widetilde{H}(y,t)) \leq f(y) - \sigma t$, where $\widetilde{B}_{2\widetilde{\delta}}(x) \subset B_{\delta}(x)$.

For $\widetilde{s} = \sup\{s \in [0,1] |\ \alpha(s) \in B_{3\delta}(x), \alpha \in \Gamma_{y,\widetilde{H}(y,t)}\}$, it follows that

$$\begin{aligned}\int_0^{\widetilde{s}} \beta(\rho(\alpha(s),A)) \|\alpha'(s)\| ds &\geq (\beta(\rho(x,A)) - \varepsilon) \|\alpha(\widetilde{s}) - \frac{1}{C} y\| \\ &\geq (\beta(\rho(x,A)) - \varepsilon) \|\widetilde{H}(y,t) - \frac{1}{C} y\|.\end{aligned}$$

This inequality implies

$$(\beta(\rho(x,A)) - \varepsilon) \|\widetilde{H}(y,t) - \frac{1}{C} y\| \leq \widetilde{\rho}(\widetilde{H}(y,t), y) \leq t.$$

Now we can take $\overline{\delta} > 0$ such that $(\beta(\rho(x,A)) - \varepsilon)\overline{\delta} \leq \widetilde{\delta}$ and $B_{\overline{\delta}}(x) \subset \widetilde{B}_{\widetilde{\delta}}(x)$ and we define $\mathcal{H} : B_{\overline{\delta}}(x) \times [0, \overline{\delta}] \to X$ by

$$\mathcal{H}(y,t) = \widetilde{H}(y, (\beta(\rho(x,A)) - \varepsilon)t.$$

Then $\|\widetilde{H}(y,t) - \frac{1}{C} y\| \leq t$ and $f(H(y,t)) \leq f(y) - \sigma(\beta(\rho(x,A)) - \varepsilon)t$, which shows that $|\widetilde{df}|(x) \geq \sigma(\beta(\rho(x,A)) - \varepsilon)$ and consequently $|\widetilde{df}|(x) \geq \sigma\beta(\rho(x,A))$.

(ii) We follow [10, Theorem 2.2] and the same rule as in (i). $\square$

6. Deformation theorems

Useful tools in classical critical point theory are the well known deformation lemmas; see Palais [17], Chang [4]. For the case of metric spaces, the problem was studied by Corvellec, Degiovanni and Marzocchi in [8], [9].

In the context of quasi-metric spaces, the deformation theorems have the following formulation:

Theorem 6.1. *Let (X,ρ) be a quasi-metric space. Let $f : X \to \mathbb{R}$ and $\sigma : X \to [0,\infty)$ be continuous such that $|df|(x) \neq 0$ implies $|df|(x) > \sigma(x) > 0$. Then there exist $\tau : X \to [0,\infty)$ and $\eta : X \times [0,\infty) \to X$ continuous such that the following properties are satisfied:*

(i) $\tau(x) = 0$ if and only if $|df|(x) = 0$;

(ii) $\rho(\eta(x,t), x) \leq t$;

(iii) if $\tau(x) \leq t$, then $\eta(x,t) = \eta(x, \tau(x))$;

(iv) if $t \leq \tau(x)$, then $f(\eta(x,t)) \leq f(x) - \sigma(x)t$.

In particular, $\eta(x,t) = x$ *if and only if* $|df|(x) = 0$.

Theorem 6.2. *Let* (X, ρ) *be a quasi-metric space. Let* $f : X \to \mathbb{R}$ *be a continuous function such that the following assumptions are satisfied:*

(i) *for any* $a < c < b$, *the set* $f^{-1}[a, c]$ *is complete;*

(ii) *for any* $a < c < b$, f *satisfies the* (PS) *condition on* $f^{-1}[a, c]$;

(iii) $K[f] \cap f^{-1}(a, b) = \emptyset$;

(iv) $K_a[f] = \emptyset$ *or the connected components of* $K_a[f]$ *are single points.*

Then $X_a[f] \cup K_a[f]$ *is a deformation retract of* $X_b[f]$.

For the proofs we use Theorem 5.2 and follow step by step [8, Theorem 1] and [8, Theorem 4].

7. Stability of critical values

An interesting problem is the stability under perturbation of ρ-critical values; more precisely, we want to know if two enough close continuous functions on a quasi-metric space have close critical values.

This problem was first analyzed by Marino and Prodi in [15] for C^2-functions on complete Riemann manifolds. The G-Riemann-Hilbert case, where G is a compact Lie group, is studied in [2]. The case of Finsler manifolds appears in [18] respectively in [3] and the metric case in [5].

We denote by $H_q(B, A)$ the q^{th} relative singular homology group of the pair (B, A) with real coefficients, where $A \subset B$ and q is a nonnegative integer. $H_*(B, A)$ denotes the graded group $(H_q(B, A))_q$.

We mention that for a deformation retract A' of A we have $H_n(A, A') = 0$, $\forall\, n$. Moreover, if $A'' \subset A' \subset A$ and A' is a deformation retract of A, then, for any nonnegative integer n, we have $H_n(A, A'') = H_n(A', A'')$. (See, for instance, [16].)

We need the following lemma (see [15]):

Lemma 7.1. *Let* A, X, B, A', Y, B' *be topological spaces such that* $A \subset X \subset B \subset A' \subset Y \subset B'$. *Assume that* $H_n(B, A) = 0$ *and* $H_n(B', A') = 0$, *for any* n. *Then there exist an injective homomorphism* $h : H_n(A', A) \to H_n(Y, X)$.

Theorem 7.2. *Let* (X, ρ) *be a quasi-metric space. Let* $f, g : X \to \mathbb{R}$ *continuous functions such that* $c \in \mathbb{R}$ *is the only* ρ*-critical value of* f *in* $[c - \varepsilon, c + \varepsilon]$, *where* $\varepsilon > 0$. *Assume that for any* x *in* $[c - \varepsilon, c + \varepsilon)$, *the set* $f^{-1}[c - \varepsilon, x]$ *is complete,* f *satisfies the* (PS)*-condition on* $f^{-1}[c - \varepsilon, x]$ *and* g *satisfies the* (PS)*-condition on* $g^{-1}[c - \varepsilon, x]$.

If there exist m *such that* $H_m(X_{c+\varepsilon}[f], X_{c-\varepsilon}[f]) \neq 0$ *and* $\delta = \delta_\varepsilon > 0$ *such that*

$$|f(x) - g(x)| \leq \delta, \ \forall x \in X,$$

then there exists a ρ*-critical value of* g *in the interval* $[c-(\varepsilon-\delta), c+(\varepsilon-\delta)]$.

Proof. We follow the idea of [15, Teorema 4.1] and [5, Theorem 4.1].

If $\varepsilon > 2\delta$, the inequality $|f(x) - g(x)| \leq \delta$ implies

$$X_{c-\varepsilon}[f] \subset X_{c-(\varepsilon-\delta)}[g] \subset X_{c-(\varepsilon-2\delta)}[f] \subset X_{c+(\varepsilon-2\delta)}[f] \subset X_{c+(\varepsilon-\delta)}[g] \subset X_{c+\varepsilon}[f].$$

Because $[c-\varepsilon, c+\varepsilon] \cap f(K[f]) = \{c\}$, Theorem 6.2 shows that $X_{c-\varepsilon}[f]$ is a deformation retract of $X_{c-(\varepsilon-2\delta)}[f]$ and $X_{c+(\varepsilon-2\delta)}[f]$ is a deformation retract of $X_{c+\varepsilon}[f]$. Then $H_n(X_{c-(\varepsilon-2\delta)}[f], X_{c-\varepsilon}[f]) = H_n(X_{c+\varepsilon}[f], X_{c+(\varepsilon-2\delta)}[f]) = 0$, for any n. Apply Lemma 7.1 and find an injective homomorphism

$$h : H_n(X_{c+(\varepsilon-2\delta)}[f], X_{c-\varepsilon}[f]) \to H_n(X_{c+(\varepsilon-\delta)}[g], X_{c-(\varepsilon-\delta)}[g]).$$

On the other hand,

$$H_n(X_{c+(\varepsilon-2\delta)}[f], X_{c-\varepsilon}[f]) = H_n(X_{c+\varepsilon}[f], X_{c-\varepsilon}[f]),$$

for any n, $X_{c+(\varepsilon-2\delta)}[f]$ being a deformation retract of $X_{c+\varepsilon}[f]$.

Now, we can use the assumption $H_m(X_{c+\varepsilon}[f], X_{c-\varepsilon}[f]) \neq 0$ and it follows that

$$H_m(X_{c+(\varepsilon-\delta)}[g], X_{c-(\varepsilon-\delta)}[g]) \neq 0$$

and $[c-(\varepsilon-\delta), c+(\varepsilon-\delta)] \cap g(K[g]) \neq \emptyset$. $\square$

Remark that the above theorem assures the existence of at least one ρ-critical point of g. Moreover, if f has p ρ-critical values, in the corresponding hypothesis, g has at least p ρ-critical points.

The assumption $|f(x) - g(x)| \leq \delta = \dfrac{\varepsilon}{M}, \forall x \in X$, where $M > 2$, implies $X_{c-\varepsilon}[f] \subset X_{c-(\varepsilon-\delta)}[g] \subset X_{c-\frac{\varepsilon}{2}}[g] \subset X_{c-(\frac{\varepsilon}{2}-\delta)}[f] \subset X_{c+(\frac{\varepsilon}{2}-\delta)}[f] \subset X_{c+\frac{\varepsilon}{2}}[g] \subset X_{c+(\varepsilon-\delta)}[g] \subset X_{c+\varepsilon}[f]$. Then the shortest interval which appears in the conclusion of Theorem 7.2 is $[c-\dfrac{\varepsilon}{2}, c+\dfrac{\varepsilon}{2}]$.

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