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INVARIANT APPROXIMATIONS, NONCOMMUTING, GENERALIZED $\mathcal{I}$-NONEXPANSIVE MAPPINGS AND NON-STARSHAPED SET IN q-NORMED SPACE

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Abstract. We establish common fixed point results for noncommuting nonlinear generalized $\mathcal{I}$-nonexpansive mappings in the setup of nonstarshaped set of q-normed space. As application, various invariant approximation results are also obtained. Our results inprove, extend and generalize various existing known results in the literature.

1. Introduction and preliminaries

Let X be a linear space. A q-norm on X is a real-valued function $\|.\|_q$ on X with $0 < q \leq 1$, satisfying the following conditions :

(a) $\|x\|_q \geq 0$ and $\|x\|_q = 0$ iff $x = 0$,
(b) $\|\lambda x\|_q = |\ \lambda\ |^q \|x\|_q$,
(c) $\|x + y\|_q \leq \|x\|_q + \|y\|_q$,

for all $x, y \in X$ and all scalars λ. The pair $(X, \|.\|_q)$ is called a q-normed space. It is a metric space with $d_q(x, y) = \|x - y\|_q$ for all $x, y \in X$, defining a translation invariant metric d_q on X. If $q = 1$, we obtain the concept of a normed linear space. It is well-known that the topology of every Housdorff locally bounded topological linear space is given by some q-norm, $0 < q \leq 1$. The spaces l_q and $L_q[0, 1]$, $0 < q \leq 1$ are q-normed space. A q-normed space is not necessarily a locally convex space. Recall that, if X is a topological

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linear space, then its continuous dual space X^* is said to separate the points of X, if for each $x \neq 0$ in X, there exists an $\mathcal{I} \in X^*$ such that $\mathcal{I}x \neq 0$. In this case the weak topology on X is well-defined. We mention that, if X is not locally convex, then X^* need not separates the points of X. For example, if $X = L_q[0,1]$, $0 < q < 1$, then $X^* = \{0\}$ ([15], page 36 and 37). However, there are some non-locally convex spaces (such as the q-normed space l_q, $0 < q < 1$) whose dual separates the points [10].

Let X be a metric space and let C be a nonempty subset of X. Let $x \in X$. An element $y \in C$ is called a best C-approximant to $x \in X$ if

$$dist(x, C) = \inf\{d(x, z) : z \in C\}.$$

The set of best C-approximants to x is denoted by $P_C(x_0)$ and is defined as $P_C(x_0) = \{y \in C : d(x, y) = dist(x, C)\}$. The map $\mathcal{T} : C \to X$ is said to be completely continuous if $\{x_n\}$ converges weakly to x implies that $\{\mathcal{T}x_n\}$ converges strongly to $\mathcal{T}x$. Let $\mathcal{I} : C \to C$ be a mapping. A mapping $\mathcal{T} : C \to C$ is called an $\mathcal{I}$-contraction if there exists $0 \leq k < 1$ such that $d(\mathcal{T}x, \mathcal{T}y) \leq kd(\mathcal{I}x, \mathcal{I}y)$ for any $x, y \in C$. If $k = 1$, then $\mathcal{T}$ is called $\mathcal{I}$-nonexpansive. The set of fixed points of $\mathcal{T}$(resp. $\mathcal{I}$) is denoted by $F(\mathcal{T})$ (resp. $F(\mathcal{I})$) and closure of $\mathcal{T}$ by $cl(\mathcal{T})$. A point $x \in C$ is a common fixed point of $\mathcal{I}$ and $\mathcal{T}$ if $x = \mathcal{I}x = \mathcal{T}x$. The pair $(\mathcal{I}, \mathcal{T})$ is called (1) commuting if $\mathcal{I}\mathcal{T}x = \mathcal{T}\mathcal{I}x$ for all $x \in C$; (2) $R-$weakly commuting if for all $x \in C$ there exists $R > 0$ such that $d(\mathcal{T}\mathcal{I}x, \mathcal{I}\mathcal{T}x) \leq Rd(\mathcal{T}x, \mathcal{I}x)$. If $R = 1$, then the maps are called weakly commuting. The set C is $p-$starshaped with $p \in F(\mathcal{I})$ if the segment $[p, x] = \{(1-k)p + kx\}$ joining p to x, is contained in C for all $x \in C$. Suppose C is $p-$starshaped with $p \in F(\mathcal{I})$ and is both $\mathcal{T}-$ and $\mathcal{I}-$ invariant. Then $\mathcal{T}$ and $\mathcal{I}$ are called R-subweakly commuting on C [18] if there exists $R \in (0, \infty)$ such that $d(\mathcal{T}\mathcal{I}x - \mathcal{I}\mathcal{T}x) \leq R \; dist(\mathcal{I}x, [\mathcal{T}x, p])$ for all $x \in C$. It is well-known that commuting maps are R-subweakly commuting maps and R-subweakly commuting maps are R-weakly commuting but not conversely in general (see [17, 18]).

We give the definition providing the notion of contractive jointly continuous family introduced by Dotson [4].

Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in X}$ a family of functions from $[0,1]$ into C such that $f_\alpha(1) = \alpha$ for each $\alpha \in C$. The family $\mathcal{F}$ is said to be contractive, if there exists a function $\phi : (0,1) \to (0,1)$ such that for all $\alpha, \beta \in C$ and all $t \in (0,1)$, we have

$$d(f_\alpha(t) - f_\beta(t)) \leq \phi(t) d(\alpha - \beta).$$

The family $\mathcal{F}$ is said to be jointly continuous(resp. jointly weakly continuous) if $t \to t_0$ in $[0,1]$ and $\alpha \to \alpha_0$ in X(resp. if $t \to t_0$ in $[0,1]$ and $\alpha \to^w \alpha_0$ in C), then $f_\alpha(t) \to f_{\alpha_0}(t_0)$ (resp. $f_\alpha(t) \to^w f_{\alpha_0}(t_0)$) in X; here $\to$ and $\to^w$ denotes the strong and weak convergence respectively.

Now, we give the *property* (Γ) on contractive jointly continuous family $\mathcal{F}$.

A self mapping $\mathcal{T}$ of C is said to satisfy the *property* (Γ), if for any $t \in [0,1]$, for all $x \in C$ and for all $f_x \in \mathcal{F}$, we have $\mathcal{T}(f_x(t)) = f_{\mathcal{T}x}(t)$.

During the last four decades several interesting and valuable results as application of fixed point theorems were studied extensively in the field of invariant approximation theory.

In 1963, Meinardus [11] was the first who observed the general principle and employed a fixed point theorem to establish the existence of an invariant approximation. Afterwards in 1969, Brosowski [2] obtained the following generalization of Meinardus's result.

Theorem 1.1. *Let X be a normed space and $\mathcal{T} : X \to X$ be a linear and nonexpansive operator. Let C be a $\mathcal{T}-$invariant subset of X and $x_0 \in F(\mathcal{T})$. If $P_C(x_0)$, the set of best approximants of x_0 in C, is nonempty compact and convex, then there exists a y in $P_C(x_0)$ which is also a fixed point of $\mathcal{T}$.*

On the other hand, Subrahmanyam [23] obtained the following generalization of the above mentioned theorem of Meinardus [11].

Theorem 1.2. *Let X be a normed space. If $\mathcal{T} : X \to X$ is a nonexpansive operator with a fixed point x_0, leaving a finite dimensional subspace C of X invariant, then there exists a best approximant of x_0 in C which is also a fixed point of $\mathcal{T}$.*

In 1981, Smoluk [22] observed that the finite dimensionality in the result of Subrahmanyam can be replaced by the linearity and compactness of $\mathcal{T}$. Later, Habiniak [5] noted that the linearity of $\mathcal{T}$ in Smoluk′s result is redundant.

In 1979, Singh [19] observed that the linearity of mapping $\mathcal{T}$ and the convexity of the set $P_C(x_0)$ of best approximant of x_0 in Theorem 1.1, can be relaxed and proved the following extension of it.

Theorem 1.3. *Let X be a normed space, $\mathcal{T} : X \to X$ be a nonexpansive mapping, C be a $\mathcal{T}-$invariant subset of X and $x_0 \in F(\mathcal{T})$. If $P_C(x_0)$ is nonempty compact and starshaped, then there exists a best approximant of x_0 in C which is also a fixed point of $\mathcal{T}$.*

In a subsequent paper, Singh [20] also observed that only the nonexpansiveness of $\mathcal{T}$ on $P_C(x_0) \cup \{x_0\}$ is necessary for the validity of Theorem 1.3. Further in 1982, Hicks and Humpheries [6] had shown that Theorem 1.3 remain true, if $\mathcal{T} : C \to C$ is replaced by $\mathcal{T} : \partial C \to C$, where ∂C, denotes the boundary of C. Furthermore, Sahab, Khan and Sessa [16] generalized the result of Hicks and Humpheries [6]and Theorem 1.3 using two mappings, one linear and other nonexpansive for commuting mappings and established the following result of common fixed point for best approximant in setup of normed space. They took this idea from Park [13].

Theorem 1.4. *Let $\mathcal{I}$ and $\mathcal{T}$ be self maps of X with $x_0 \in F(\mathcal{I}) \cap F(\mathcal{T})$, $C \subset X$ with $\mathcal{T} : \partial C \to C$, and $p \in F(\mathcal{I})$. If $P_C(x_0)$, the set of best approximant is compact and $p-$starshaped, $\mathcal{I}(P_C(x_0)) = P_C(x_0)$, $\mathcal{I}$ is continuous and linear on $P_C(x_0)$, $\mathcal{I}$ and $\mathcal{T}$ are commuting on $P_C(x_0)$ and $\mathcal{T}$ is $\mathcal{I}-$nonexpansive on $P_C(x_0) \cup \{x_0\}$, then $\mathcal{I}$ and $\mathcal{T}$ have a common fixed point in $P_C(x_0)$.*

Recently, Al-Thagafi [1] generalized result of Sahab, Khan and Sessa [16] and proved some results on invariant approximations for commuting mappings. More recently, with the introduction of non-commuting maps to this area, O'Regan and Shahzad [14], and Shahzad [17, 18] further extended Al-Thagafi's results and obtained a numbere of results regarding invariant approximation. All the above mentioned results are obtained on starshaped domain and linearity or affinness condition of mapping.

Here it is important to remark that Dotson [3] proved the existence of fixed point for nonexpansive mapping in the setup of starshaped. He further extended his result without starshapedness under non-convex condition [4]. This idea was utilized by Mukherjee and Som [12] to prove existence of fixed point as best approximant. In this way, they extended the result of Singh [19] without starshapedness condition. In a paper, Khan and Khan [7] extended a fixed point theorem of Dotson [4] and generalized a invariant approximation result of Smoluk [22] in the setting of q- normed space. Further, Khan, Hussain and Thaheem [8] extended the results of Khan and Khan [7] and generalized the result of Singh [19] by using the concept of nonconvexity of Dotson [4]. More recently, Khan, Latif, Bano and Hussain [9] proved some results on invariant approximations for commuting mappings in non-starshaped set of q-normed space and extended and generalized the results of Al-Thagafi [1], Habiniak [5], Khan, Hussain and Thaheem [8], Sahab, Khan and Sessa [16] and Singh [19].

The purpose of this paper is to find existance results on common fixed point for noncommuting nonlinear generalized $\mathcal{I}$-nonexpansive mappings to a domain which is not necessarily starshaped in q-normed space. As application, various invariant approximation result are also obtained. Our results improve, extend, generalize, and compliment those of Al-Thagafi [1], Dotson [3, 4], Habiniak [5], Khan and Khan [7], Khan, Hussain and Thaheem [8], Khan, Latif, Bano and Hussain [9], Mukherjee and Som [12], O'Regan and Shahzad [14], Sahab, Khan and Sessa [16], Shahzad [17, 18] and Singh [19].

2. Main results

The following result of O'Regan and Shahzad [14] is needed in the sequel:

Theorem 2.1 ([14]). *Let C be a closed subset of a metric space (X, d) and $\mathcal{T}$ and $\mathcal{I}$ be R-weakly commuting self-maps of C such that $\mathcal{T}(C) \subseteq \mathcal{I}(C)$. Suppose there exists $\lambda \in (0,1)$ such that*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \lambda \ max\{d(\mathcal{I}x, \mathcal{I}y), d(\mathcal{T}x, \mathcal{I}x), d(\mathcal{T}y, \mathcal{I}y), d(\mathcal{T}x, \mathcal{I}y), d(\mathcal{T}y, \mathcal{I}x)]\}$$

for all $x, y \in C$. If $cl(\mathcal{T}(C))$ is complete and either $\mathcal{T}$ or $\mathcal{I}$ is continuous, then $F(\mathcal{T}) \cap F(\mathcal{I}) \cap C$ is singleton.

We first prove common fixed theorem for noncommuting nonlinear generalized $\mathcal{I}$-nonexpansive mappings in the set which is not necessarily starshaped of q-normed space.

Theorem 2.2. *Let C be a subset of q-normed space X and $\mathcal{T}$ and $\mathcal{I}$ be self-mappings of C such that $\mathcal{T}(C) \subset \mathcal{I}(C)$. Suppose C has a contractive family of functions $\mathcal{F} = \{f_\alpha\}_{\alpha \in C}$. If $\mathcal{I}$ satisfies the property (Γ) on C, $\mathcal{T}$ and $\mathcal{I}$ satisfy*

$$\|\mathcal{T}\mathcal{I}x - \mathcal{I}\mathcal{T}x\|_q \leq R \ \|f_{\mathcal{T}x}(k) - \mathcal{I}x\|_q \tag{2.1}$$

for all $x \in C$, $R > 0$, and

$$\begin{aligned}\|\mathcal{T}x - \mathcal{T}y\|_q \leq \max\{&\|\mathcal{I}x - \mathcal{I}y\|_q, dist(f_{\mathcal{T}x}(k), \mathcal{I}x), dist(f_{\mathcal{T}y}(k), \mathcal{I}y), \\ &dist(f_{\mathcal{T}y}(k), \mathcal{I}x), dist(f_{\mathcal{T}x}(k), \mathcal{I}y)\}\end{aligned} \tag{2.2}$$

for all $x \neq y \in C$, $k \in (0,1)$, then $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$, provided one of the following conditions holds:

- *(i) C is closed, $cl(\mathcal{T}(C))$ is compact, $\mathcal{I}$ and $\mathcal{T}$ are continuous, and family $\mathcal{F}$ is jointly continuous,*
- *(ii) X is complete with separating dual X^*, C is weakly compact, $\mathcal{I}$ and $\mathcal{T}$ are weakly continuous, and family $\mathcal{F}$ is weakly jointly continuous,*
- *(iii) X is complete with separating dual X^*, C is weakly compact, $\mathcal{T}$ is completely continuous, $\mathcal{I}$ is continuous, and family $\mathcal{F}$ is jointly continuous,*
- *(iv) X is complete with separating dual X^*, C is weakly compact, $\mathcal{I}$ is demicompact, $\mathcal{T}$ and $\mathcal{I}$ are continuous, and family $\mathcal{F}$ is jointly continuous.*

Proof. Choose a sequence $k_n \in (0,1)$ with $\{k_n\} \to 1$ as $n \to \infty$. Define for each $n \geq 1$ and for all $x \in C$, a mapping $\mathcal{T}_n$ by

$$\mathcal{T}_n x = f_{\mathcal{T}x}(k_n).$$

Then, each $\mathcal{T}_n$ is well defined from C into C for each n and $\mathcal{T}_n(C) \subset \mathcal{I}(C)$ for each n, since $\mathcal{T}(C) \subset \mathcal{I}(C)$. The *property* (Γ) of $\mathcal{I}$ and (2.1) imply that

$$\begin{aligned}
\|\mathcal{T}_n\mathcal{I}x - \mathcal{I}\mathcal{T}_n x\|_q &= \|f_{\mathcal{T}\mathcal{I}x}(k_n) - \mathcal{I}f_{\mathcal{T}x}(k_n)\|_q \\
&= \|f_{\mathcal{T}\mathcal{I}x}(k_n) - f_{\mathcal{I}\mathcal{T}x}(k_n)\|_q \\
&\leq [\phi(k_n)]^q \|\mathcal{T}\mathcal{I}x - \mathcal{I}\mathcal{T}x\|_q \\
&\leq [\phi(k_n)]^q R \|f_{\mathcal{T}x}(k_n) - \mathcal{I}x\|_q \\
&= [\phi(k_n)]^q R \|\mathcal{T}_n x - \mathcal{I}x\|_q
\end{aligned}$$

for all $x \in C$. Thus $\mathcal{T}_n$ and $\mathcal{I}$ are $[\phi(k_n)]^q R-$ weakly commuting. Also by (2.2),

$$\begin{aligned}
\|\mathcal{T}_n x - \mathcal{T}_n y\|_q &= \|f_{\mathcal{T}x}(k_n) - f_{\mathcal{T}y}(k_n)\|_q \\
&\leq [\phi(k_n)]^q \|\mathcal{T}x - \mathcal{T}y\|_q \\
&\leq [\phi(k_n)]^q \max\{\|\mathcal{I}x - \mathcal{I}y\|_q, dist(f_{\mathcal{T}x}(k_n), \mathcal{I}x), dist(f_{\mathcal{T}y}(k_n), \mathcal{I}y), \\
&\qquad dist(f_{\mathcal{T}y}(k_n), \mathcal{I}x), dist(f_{\mathcal{T}x}(k_n), \mathcal{I}y)\} \\
&\leq [\phi(k_n)]^q \max\{\|\mathcal{I}x - \mathcal{I}y\|_q, \|\mathcal{T}_n x - \mathcal{I}x\|_q, \|\mathcal{T}_n y - \mathcal{I}y\|_q, \\
&\qquad \|\mathcal{T}_n y - \mathcal{I}x\|_q, \|\mathcal{T}_n x - \mathcal{I}y\|_q\}
\end{aligned}$$

for all $x, y \in C$.

(i) Since $cl(\mathcal{T}(C))$ is compact, $cl(\mathcal{T}_n(C))$ is also compact. It follows from Theorem 2.1, for each $n \geq 1$, there exists $x_n \in C$ such that $x_n = \mathcal{T}_n x_n = \mathcal{I}x_n$. As $cl(\mathcal{T}(C))$ is compact and $\{\mathcal{T}x_n\}$ is sequence in it, so $\{\mathcal{T}x_n\}$ has a subsequence $\{\mathcal{T}x_m\}$ converging, e.g., to $y \in cl(\mathcal{T}(C))$.

$$x_m = \mathcal{T}_m x_m = f_{\mathcal{T}x_m}(k_m)$$

converges to y. By the continuity of $\mathcal{T}$, $\{\mathcal{T}x_m\}$ converges to $\mathcal{T}y$. But $\mathcal{T}x_m$ tends to y by the assumption. So, by the jointly continuity of $\mathcal{F}$, we have

$$\mathcal{T}_m x_m = f_{\mathcal{T}x_m}(k_m) \to f_{\mathcal{T}y}(1) = \mathcal{T}y, \ \ as \ \ m \to \infty$$

Thus, $\mathcal{T}y = y$. Also from the continuity of $\mathcal{I}$, we have

$$\mathcal{I}y = \mathcal{I}(\lim x_m) = \lim \mathcal{I}x_m = \lim x_m = y, \ \ as \ \ m \to \infty$$

i.e., $\mathcal{I}y = y$. Hence $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.

(ii) As in (i) there exists $x_n \in C$ such that $x_n = \mathcal{T}_n x_n = \mathcal{I}x_n$. Since C is weakly compact, there is a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to some $w \in C$ as $m \to \infty$. But, $\mathcal{I}$ is weakly continuous so we have $\mathcal{I}w = w$. Now, $\mathcal{T}x_m \to^w \mathcal{T}w$ and hence $x_m = f_{\mathcal{T}x_m}(k_m) \to f_{\mathcal{T}w}(1) = \mathcal{T}w$. Also since $x_m \to w$ and the weak topology is Hausdorff, we get $\mathcal{T}w = w$. Hence $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.

(iii) As in (i) there exists $x_n \in C$ such that $x_n = \mathcal{T}_n x_n = \mathcal{I}x_n$. Since C is weakly compact, there is a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to some $y \in C$ as $m \to \infty$. Since $\mathcal{T}$ is completely continuous, $\mathcal{T}x_m \to \mathcal{T}y$ as $m \to \infty$. Now using the joint continuity of $\mathcal{F}$ and $k_m \to 1$, we get

$$x_m = f_{\mathcal{T}x_m}(k_m) \to f_{\mathcal{T}y}(1) = \mathcal{T}y \ \ as \ \ m \to \infty.$$

Thus $\mathcal{T}x_m \to \mathcal{T}^2 y$ as $m \to \infty$ and consequently $\mathcal{T}^2 y = \mathcal{T}y$ implies that $\mathcal{T}z = z$, where $z = \mathcal{T}y$. But, since $\mathcal{I}x_m = x_m \to \mathcal{T}y = z$, using the continuity of $\mathcal{I}$ and the uniqueness of the limit, we have $\mathcal{I}z = z$. Hence $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.

(iv) Suppose that (x_n) is a bounded sequence and $(\mathcal{T}x_n - x_n)$ converges strongly to 0. By demicompactness of $\mathcal{I}$, (x_n) has a subsequence (x_m) converges strongly to x(say) in C and hence $x_m = \mathcal{T}x_m \to \mathcal{I}x$ implies that $x = \mathcal{I}x$. Also $\mathcal{T}x_m \to \mathcal{T}x$. Further, from the joint continuity of $\mathcal{F}$ and $k_m \to 1$, we have $x_m = f_{\mathcal{T}x_m}(k_m) \to f_{\mathcal{T}x}(1) = \mathcal{T}x \ \ as \ \ m \to \infty$. Since the strong topology is Hausdorff, we get $\mathcal{T}x = x$. Hence $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$. This completes the proof. □

Immediate consequences of the Theorem 2.2 are as follows:

Corollary 2.3. *Let C be a subset of q-normed space X, and $\mathcal{T}$ and $\mathcal{I}$ self-mappings of C such that $\mathcal{T}(C) \subset \mathcal{I}(C)$. Suppose C has a contractive family $\mathcal{F} = \{f_\alpha\}_{\alpha \in C}$. If $\mathcal{I}$ satisfies the property* (Γ) *on C, $\mathcal{T}$ and $\mathcal{I}$ satisfy (2.1) for $R > 0$ and $\mathcal{T}$ is $\mathcal{I}$-nonexpansive on C, then $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$ under each of the conditions of Theorem 2.2.*

Corollary 2.4. *Let C be a subset of q-normed space X, and $\mathcal{T}$ and $\mathcal{I}$ self-mappings of C such that $\mathcal{T}(C) \subset \mathcal{I}(C)$. Suppose C has a contractive family $\mathcal{F} = \{f_\alpha\}_{\alpha \in C}$ and $\mathcal{I}$ satisfies the property* (Γ) *on C. If $\mathcal{T}$ and $\mathcal{I}$ are commutative and satisfy (2.2) for all $x, y \in C$, $k \in (0,1)$, then $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$ under each of the conditions of Theorem 2.2.*

Remark 2.5. In the light of the comment given by Dotson [4] and Khan, Latif, Bano and Hussain [9] that if $C \subseteq X$ is $p-$starshaped and $f_\alpha(t) = (1-t)p+t\alpha$, $(\alpha \in C, t \in [0,1])$, then $\{f_\alpha\}_{\alpha \in C}$ is a contractive jointly continuous family with $\phi(t) = t$. Thus the class of subsets of X with the property of contractiveness and jointly continuity contains the class of starshaped sets which in turns contains the class of convex sets. If for a subset C of X, there exists a contractive jointly continuous family $\mathcal{F} = \{f_\alpha\}_{\alpha \in C}$, then we say that C has the property of contractiveness and joint continuity.

Corollary 2.6. *Let C be a subset of q-normed space X, and $\mathcal{T}$ and $\mathcal{I}$ self-mappings of C such that $\mathcal{T}(C) \subset \mathcal{I}(C)$. Suppose C is p-starshaped, and $\mathcal{I}$ is affine with $p \in F(\mathcal{I})$. If $\mathcal{T}$ and $\mathcal{I}$ are R-subweakly commuting and satisfy, for all $x, y \in C$,*

$$\begin{aligned} \|\mathcal{T}x - \mathcal{T}y\|_q \leq \max\{&\|\mathcal{I}x - \mathcal{I}y\|_q, dist([\mathcal{T}x, p], \mathcal{I}x), dist([\mathcal{T}y, p], \mathcal{I}y), \\ &dist([\mathcal{T}y, p], \mathcal{I}x), dist([\mathcal{T}x, p], \mathcal{I}y)\} \end{aligned} \tag{2.3}$$

then $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$ provided one of the following conditions holds:

(i) C is closed, $cl(\mathcal{T}(C))$ is compact, and pair $\{\mathcal{T}, \mathcal{I}\}$ is continuous,

(ii) X is complete with separating dual X^*, C is weakly compact, and pair $\{\mathcal{T}, \mathcal{I}\}$ is weakly continuous,

(iii) X is complete with separating dual X^*, C is weakly compact, $\mathcal{T}$ is completely continuous, and $\mathcal{I}$ is continuous,

(iv) X is complete with separating dual X^*, C is weakly compact, $\mathcal{I}$ is demicompact, and pair $\{\mathcal{T}, \mathcal{I}\}$ is continuous.

Corollary 2.7. *Let C be a subset of q-normed space X, and $\mathcal{T}$ and $\mathcal{I}$ self-mappings of C such that $\mathcal{T}(C) \subset \mathcal{I}(C)$. Suppose C is p-starshaped, and $\mathcal{I}$ is affine with $p \in F(\mathcal{I})$. If $\mathcal{T}$ and $\mathcal{I}$ are R-subweakly commuting and satisfy, for all $x, y \in C$,*

$$\begin{aligned} \|\mathcal{T}x - \mathcal{T}y\|_q \leq \max\{&\|\mathcal{I}x - \mathcal{I}y\|_q, dist([\mathcal{T}x, p], \mathcal{I}x), dist([\mathcal{T}y, p], \mathcal{I}y), \\ &\tfrac{1}{2}[dist([\mathcal{T}y, p], \mathcal{I}x) + dist([\mathcal{T}x, p], \mathcal{I}y)]\} \end{aligned} \tag{2.4}$$

then $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$ under each of the conditions of Theorem 2.6.

Corollary 2.8. *Let C be a subset of q-normed space X, and $\mathcal{T}$ and $\mathcal{I}$ self-mappings of C such that $\mathcal{T}(C) \subset \mathcal{I}(C)$. Suppose C is p-starshaped, and $\mathcal{I}$ is affine with $p \in F(\mathcal{I})$. If $\mathcal{T}$ and $\mathcal{I}$ are R-subweakly commuting and $\mathcal{T}$ is $\mathcal{I}$-nonexpansive on C, then $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$ under each of the conditions of Theorem 2.6.*

Corollary 2.9. *Let C be a subset of q-normed space X, and $\mathcal{T}$ and $\mathcal{I}$ self-mappings of C such that $\mathcal{T}(C) \subset \mathcal{I}(C)$. Suppose C is p-starshaped, and $\mathcal{I}$ is affine with $p \in F(\mathcal{I})$. If $\mathcal{T}$ and $\mathcal{I}$ are commuting and satisfy (2.3), for all $x, y \in C$, then $C \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$ under each of the conditions of Theorem 2.6.*

Following Al-Thafgafi [1], we define $D = P_C(x_0) \cap \mathcal{D}_C^{\mathcal{I}}(x_0)$, where $\mathcal{D}_C^{\mathcal{I}}(x_0) = \{x \in C : \mathcal{I}x \in P_C(x_0)\}$.

Theorem 2.10. *Let X be a q-normed space and $\mathcal{T},\mathcal{I} : X \to X$. Let C be subset of X such that $\mathcal{T}(\partial C \cap C) \subset C$ and $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I})$. Assume D is closed, has a contractive jointly continuous family $\mathcal{F} = \{f_\alpha\}_{\alpha \in D}$, $\mathcal{I}(D) = D$, $\mathcal{I}$ satisfies the property (Γ), pair $\{\mathcal{T},\mathcal{I}\}$ is continuous on D, and $cl(\mathcal{T}(D))$ is compact. If $\mathcal{I}$ is nonexpansive on $P_M(x_0) \cup \{x_0\}$, $\mathcal{T}$ and $\mathcal{I}$ satisfy (2.1) for $x \in D$, $R > 0$, and satisfy, for all $x \in D \cup \{x_0\}$,*

$$\|\mathcal{T}x - \mathcal{T}y\|_q \le \begin{cases} \|\mathcal{I}x - \mathcal{I}x_0\|_q, \quad \text{if} \quad y = x_0, \\ \\ \max\{\|\mathcal{I}x - \mathcal{I}y\|_q, dist(\mathcal{I}x, f_{\mathcal{T}x}(k)), dist(\mathcal{I}y, f_{\mathcal{T}y}(k)), \\ \\ dist(\mathcal{I}x, f_{\mathcal{T}y}(k)), dist(\mathcal{I}y, f_{\mathcal{T}x}(k))\}, \ \text{if } y \in D, \end{cases} \tag{2.5}$$

where $k \in (0,1)$, then $P_C(x_0) \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.

Proof. Let $x \in D$. Then, $x \in P_C(x_0)$ and hence $\|x - x_0\|_q = dist(x_0, C)$. Note that for any $k \in (0,1)$,

$$\|kx_0 + (1-k)x - x_0\|_q = (1-k)^q \|x - x_0\|_q < dist(x_0, C).$$

It follows that the line segment $\{kx_0 + (1-k)x : 0 < k < 1\}$ and the set C are disjoint. Thus x is not in the interior of C and so $x \in \partial C \cap C$. Since $\mathcal{T}(\partial C \cap C) \subset C$, $\mathcal{T}x$ must be in C. Also since $\mathcal{I}x \in P_C(x_0)$, $x_0 = \mathcal{T}x_0 = \mathcal{I}x_0$ and $\mathcal{T}$ and $\mathcal{I}$ satisfy (2.5), we have

$$\|\mathcal{T}x - x_0\|_q = \|\mathcal{T}x - \mathcal{T}x_0\|_q \le \|\mathcal{I}x - \mathcal{I}x_0\|_q = \|\mathcal{I}x - x_0\|_q = dist(x_0, C).$$

Thus, $\mathcal{T}x \in P_C(x_0)$. As $\mathcal{I}$ is nonexpansive on $P_C(x_0) \cup \{x_0\}$, we have

$$\|\mathcal{I}\mathcal{T}x - x_0\|_q \le \|\mathcal{T}x - \mathcal{T}x_0\|_q \le \|\mathcal{I}x - \mathcal{I}x_0\|_q = \|\mathcal{I}x - x_0\|_q = dist(x_0, C).$$

Thus $\mathcal{I}\mathcal{T}x \in P_C(x_0)$ and so $\mathcal{T}x \in \mathcal{D}_C^{\mathcal{I}}(x_0)$. Hence $\mathcal{T}x \in D$. Consequently, $\mathcal{T}(D) \subset D = \mathcal{I}(D)$. Now Theorem 2.2 guarantees that

$$P_C(x_0) \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi.$$

This completes the proof. □

Theorem 2.11. *Let X be a q-normed space and $\mathcal{T},\mathcal{I} : X \to X$. Let C be subset of X such that $\mathcal{T}(\partial C \cap C) \subset C$ and $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I})$. Assume D is closed, has a contractive jointly continuous family $\mathcal{F} = \{f_\alpha\}_{\alpha \in D}$, $\mathcal{I}(D) = D$, $\mathcal{I}$ satisfies the property (Γ), pair $\{\mathcal{T},\mathcal{I}\}$ is continuous on D, and $cl(\mathcal{T}(D))$ is compact. If $\mathcal{T}$ and $\mathcal{I}$ are commuting on D and satisfy (2.5) for all $x \in D \cup \{x_0\}$, $k \in (0,1)$, then $P_C(x_0) \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.*

Proof. Let $x \in D$. As in the proof of Theorem 2.10, we obtain $\mathcal{T}x \in P_C(x_0)$. Moreover, since $\mathcal{T}$ commutes with $\mathcal{I}$ on D and $\mathcal{T}$ and $\mathcal{I}$ satisfy (2.5),

$$\|\mathcal{I}\mathcal{T}x - x_0\|_q = \|\mathcal{T}\mathcal{I}x - \mathcal{T}x_0\|_q \le \|\mathcal{I}^2 x - \mathcal{I}x_0\|_q = \|\mathcal{I}^2 x - x_0\|_q = dist(x_0, C).$$

Thus $\mathcal{I}\mathcal{T}x \in P_C(x_0)$ and so $\mathcal{T}x \in \mathcal{D}_C^{\mathcal{I}}(x_0)$. Hence $\mathcal{T}x \in D$. Consequently, $\mathcal{T}(D) \subset D = \mathcal{I}(D)$. Now Theorem 2.2 guarantees that $P_C(x_0)\cap F(\mathcal{T})\cap F(\mathcal{I}) \neq \phi$. This completes the proof. $\square$

Theorem 2.12. *Let X be a q-normed space and $\mathcal{T},\mathcal{I} : X \to X$. Let C be subset of X such that $\mathcal{T}(\partial C\cap C) \subset \mathcal{I}(C) \subset C$ and $x_0 \in F(\mathcal{T})\cap F(\mathcal{I})$. Assume D is closed, has a contractive jointly continuous family $\mathcal{F} = \{f_\alpha\}_{\alpha\in D}$, $\mathcal{I}$ satisfies the property (Γ), pair $\{\mathcal{T},\mathcal{I}\}$ is continuous on D, $\mathcal{I}(C)\cap D \subset \mathcal{I}(D) \subset D$ and $cl(\mathcal{T}(D))$ is compact. If $\mathcal{T}$ and $\mathcal{I}$ are commuting on D and satisfy (2.5) for all $x \in D \cup \{x_0\}$, then $P_C(x_0) \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.*

Proof. Let $x \in D$. As in Theorem 2.11, we obtain $\mathcal{T}x \in D$, that is, $\mathcal{T}(D) \subset D$ and $x \in \partial C\cap C$ and so $\mathcal{T}(D) \subset \mathcal{T}(\partial C\cap C) \subset \mathcal{I}(C)$. Thus, we can choose $y \in C$ such that $\mathcal{T}x = \mathcal{I}y$. Because $\mathcal{I}y = \mathcal{T}x \in P_C(x_0)$, it follows that $y \in \mathcal{D}_C^{\mathcal{I}}(x_0)$. Consequently, $\mathcal{T}(D) \subset \mathcal{I}(\mathcal{D}_C^{\mathcal{I}}(x_0)) \subset P_C(x_0)$. Therefore, $\mathcal{T}(D) \subset \mathcal{I}(C) \cap D \subset \mathcal{I}(D) \subset D$. Now Theorem 2.2 guarantees that $P_C(x_0)\cap F(\mathcal{T})\cap F(\mathcal{I}) \neq \phi$. This completes the proof. $\square$

Remark 2.13. We observe that $\mathcal{I}(P_C(x_0)) \subset P_C(x_0)$ implies $P_{\mathcal{C}}(x_0) \subset \mathcal{D}_C^{\mathcal{I}}(x_0)$ and hence $D = P_C(x_0)$. Consequently, Theorem 2.10, 2.11 and 2.12 remain valid when $D = P_C(x_0)$. Hence we obtain the following results.

Corollary 2.14. *Let X be a q-normed space and $\mathcal{T},\mathcal{I} : X \to X$. Let C be subset of X such that $\mathcal{T}(\partial C \cap C) \subset C$ and $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I})$. Assume $D = P_C(x_0)$ is closed, p-starshaped with $p \in F(\mathcal{I})$, $\mathcal{I}$ is affine, $\mathcal{I}(D) = D$, and $cl(\mathcal{T}(D))$ is compact. If the pair $\{\mathcal{T},\mathcal{I}\}$ is continuous, R-subweakly commuting for $x \in D$, $R > 0$, and satisfy, for all $x, y \in D \cup \{x_0\}$,*

$$\|\mathcal{T}x-\mathcal{T}y\|_q \leq \begin{cases} \|\mathcal{I}x - \mathcal{I}x_0\|_q, \;\; \text{if} \;\; y = x_0, \\ \\ \max\{\|\mathcal{I}x - \mathcal{I}y\|_q, dist([\mathcal{T}x, p], \mathcal{I}x), dist([\mathcal{T}y, p], \mathcal{I}y), \\ \\ dist([\mathcal{T}y, p], \mathcal{I}x), dist([\mathcal{T}x, p], \mathcal{I}y)\}, \;\; \text{if } y \in D, \end{cases} \tag{2.6}$$

then $P_C(x_0) \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.

Corollary 2.15. *Let X be a q-normed space and $\mathcal{T},\mathcal{I} : X \to X$. Let C be subset of X such that $\mathcal{T}(\partial C \cap C) \subset C$ and $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I})$. Assume $D = P_C(x_0)$ is closed, p-starshaped with $p \in F(\mathcal{I})$, $\mathcal{I}$ is affine, $\mathcal{I}(D) = D$, and $cl(\mathcal{T}(D))$ is compact. If the pair $\{\mathcal{T},\mathcal{I}\}$ is continuous, R-subweakly commuting for $x \in D$, $R > 0$, and satisfy, for all $x, y \in D \cup \{x_0\}$,*

$$\|\mathcal{T}x-\mathcal{T}y\|_q \leq \begin{cases} \|\mathcal{I}x - \mathcal{I}x_0\|_q, \;\; \text{if} \;\; y = x_0, \\ \\ \max\{\|\mathcal{I}x - \mathcal{I}y\|_q, dist([\mathcal{T}x, p], \mathcal{I}x), dist([\mathcal{T}y, p], \mathcal{I}y), \\ \\ \frac{1}{2}[dist([\mathcal{T}y, p], \mathcal{I}x) + dist([\mathcal{T}x, p], \mathcal{I}y)]\}, \;\; \text{if } y \in D, \end{cases} \tag{2.7}$$

then $P_C(x_0) \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.

Corollary 2.16. *Let* X *be a* q*-normed space and* $\mathcal{T}, \mathcal{I} : X \to X$. *Let* C *be subset of* X *such that* $\mathcal{T}(\partial C \cap C) \subset C$ *and* $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I})$. *Assume* $D = P_C(x_0)$ *is closed, has a contractive jointly continuous family* $\mathcal{F} = \{f_\alpha\}_{\alpha \in D}$, $\mathcal{I}(D) = D$, $\mathcal{I}$ *satisfies the property* (Γ), *pair* $\{\mathcal{T}, \mathcal{I}\}$ *is continuous on* D, *and* $cl(\mathcal{T}(D))$ *is compact. If* $\mathcal{I}$ *is nonexpansive,* $\mathcal{T}$ *is continuous,* $\mathcal{T}$ *and* $\mathcal{I}$ *commuting for* $x \in D$, *and* $\mathcal{T}$ *and* $\mathcal{I}$ *satisfy (2.5), for all* $x \in D \cup \{x_0\}$, $k \in (0,1)$, *then* $D \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.

Corollary 2.17. *Let* X *be a* q*-normed space and* $\mathcal{T}, \mathcal{I} : X \to X$. *Let* C *be subset of* X *such that* $\mathcal{T}(\partial C \cap C) \subset C$ *and* $x_0 \in F(\mathcal{T}) \cap F(\mathcal{I})$. *Assume* $D = P_C(x_0)$ *is closed, has a contractive jointly continuous* $\mathcal{F} = \{f_\alpha\}_{\alpha \in D}$, $\mathcal{I}(D) = D$, $cl(\mathcal{T}(D))$ *is compact,* $\mathcal{I}$ *satisfies the property* (Γ), *and* $\mathcal{I}$ *is nonexpansive on* D. *If* $\mathcal{T}$ *and* $\mathcal{I}$ *satisfy (2.1) for* $x \in D$, $R > 0$, *and* $\mathcal{T}$ *is* $\mathcal{I}$*-nonexpansive, for all* $x \in D \cup \{x_0\}$, $k \in (0,1)$, *then* $D \cap F(\mathcal{T}) \cap F(\mathcal{I}) \neq \phi$.

Remark 2.18. We can obtain similar invariant approximation results from Theorem 2.10, 2.11 and 2.12, using Theorem 2.2(ii), (iii) and (iv).

Remark 2.19. With Remark 2.5, Theorem 2.2 extends and generalizes the Theorem 2.2 of O'Regan and Shahzad [14] and Theorem 2.2 of Shahzad [18] using non-linear map $\mathcal{I}$, and without R-subsweakly commuting maps in the set which is not necessarily starshaped in q-normed space.

Remark 2.20. Corollary 2.6 extends Theorem 2.2 of O'Regan and Shahzad [14] and Theorem 2.2 of Shahzad [18] to q-normed space.

Remark 2.21. With Remark 2.5, Corollary 2.4 extends and generalizes Corollary 2.4 of O'Regan and Shahzad [14] using non-linear generalized $\mathcal{I}$-nonexpansive maps in the set which is not necessarily starshaped in q-normed space.

Remark 2.22. Theorem 2.2, Corollary 2.3 and Corollary 2.4 extend and generalize the result of Dotson [3], and Habiniak [5] using non-linear non-commuting generalized $\mathcal{I}$-nonexpansive maps in the set which is not necessarily starshaped in q-normed space.

Remark 2.23. Theorem 2.2, Corollary 2.3 and Corollary 2.4 extend and generalize the result of Dotson [4] using non-commuting generalized $\mathcal{I}$-nonexpansive maps in q-normed space.

Remark 2.24. Corollary 2.6- Corollary 2.9 improve and generalize the results of Dotson [3], and Habiniak [5] using non-commuting generalized $\mathcal{I}$-nonexpansive maps in q-normed space.

Remark 2.25. With Remark 2.5, Theorem 2.2 extends and generalizes the Theorem 2.2 of Al-Thagafi [1] in the sense that the map $\mathcal{I}$ is not necessarily linear and non-commuting generalized $\mathcal{I}$-nonexpansive maps defined in a domain which is not necessarily starshaped in q-normed is used in place of commuting $\mathcal{I}$-nonexpansive maps in q-normed space.

Remark 2.26. With Remark 2.5, Theorem 2.1 - Corollary2.9 generalize Theorem 3.1 and 3.4 and Theorem 2.10 - 2.12 generalize Theorem 4 of Khan and Khan [7] using two maps, non-linear noncommting generalized $\mathcal{I}$-nonexpansive maps in the domain which is not necessarily starshaped in q-normed space.

Remark 2.27. With Remark 2.5, Theorem 2.10 - 2.12 generalize Theorem 3.2 of Khan, Hussain and Thaheem [8] using two maps and noncommuting generalized $\mathcal{I}$-nonexpansive maps.

Remark 2.28. With Remark 2.5 and 2.18, Theorem 2.10-2.12 generalize Theorem 3.7, Corollary 3.8 and Theorem 3.10 of Khan and Khan [7] using non-commuting generalized $\mathcal{I}$-nonexpansive maps.

Remark 2.29. With Remark 2.5 and 2.18, Theorem 2.10 generalizes the Corollary 2.3, Theorem 2.4 and Theorem 2.5 of Khan, Latif, Bano and Hussain [9] in the sense that the non-commuting generalized $\mathcal{I}$-nonexpansive maps is used in place of commuting $\mathcal{I}$-nonexpansive maps.

Remark 2.30. With Remark 2.5 and 2.18, Theorem 2.11 and 2.12 generalize the Corollary 2.3 and Theorem 2.4 of Khan, Latif, Bano and Hussain [9] in the sense that the generalized $\mathcal{I}$-nonexpansive maps is used in place of commuting $\mathcal{I}$-nonexpansive maps.

Remark 2.31. With Remark 2.5, Theorem 2.10 extends and generalizes Theorem 3.2 of Al-Thagafi [1] and Theorem 3 of Sahab, Khan and Sessa [16] and theorem of Singh [19] in the sense that the map $\mathcal{I}$ is not necessarily linear and non-commuting generalized $\mathcal{I}$-nonexpansive maps defined in a domain which is not necessarily starshaped in q-normed space is used in place of relatively nonexpansive commutative maps.

Remark 2.32. With Remark 2.5, Theorem 2.11 and 2.12 extend and generalize Theorem 3.2 of Al-Thagafi [1], Theorem 3 of Sahab, Khan and Sessa [16] and theorem of Singh [19, 20, 21] in the sense that the map $\mathcal{I}$ is not necessarily linear and generalized $\mathcal{I}$-nonexpansive maps defined in a domain which is not necessarily starshaped in q-normed space is used in place of linear $\mathcal{I}$-nonexpansive maps.

Remark 2.33. Our Theorem 2.10 - Cor 2.17 generalize Theorem 2 of Mukherjee and Som [12] in the sense that generalized $\mathcal{I}$-nonexpansive mappings is used in place of nonexpansive mapping in q-normed space.

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