Nonlinear Functional Analysis and Applications
Vol. 12, No. 3 (2007), pp. 455-467
http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

EIGENFUNCTIONS OF THE CURL OPERATOR, ANALYTIC FUNCTIONS, AND THE HELMHOLTZ EQUATION

Konrad J. Heuvers

Department of Mathematical Sciences, Michigan Technological University,
1400 Townsend Drive, Houghton, MI 49931-1295, U.S.A.
e-mail: kheuvers@mtu.edu

Abstract. An eigenfunction $\mathbf{F}$ of the curl operator is a non-zero vector valued function satisfying $\nabla \times \mathbf{F} = \lambda \mathbf{F}$. It must also be a solution of the Helmholz equation $\nabla^2 \mathbf{F} + \lambda^2 \mathbf{F} = 0$ and must be solenoidal, i.e. $\nabla \cdot \mathbf{F} = 0$. In this paper the eigenfunctions corresponding to $\lambda \neq 0$ and having one zero component are completely determined for open convex regions. This involves arbitrary analytic functions of a complex variable. Finally, six linearly independent solutions $\mathbf{L} = \langle L_1, L_2, 0 \rangle$, $\mathbf{L}_\perp = \langle -L_2, L_1, 0 \rangle$, $\mathbf{M} = \langle 0, M_2, M_3 \rangle$, $\mathbf{M}_\perp = \langle 0, -M_3, M_2 \rangle$, $\mathbf{N} = \langle N_1, 0, N_3 \rangle$, and $\mathbf{N}_\perp = \langle N_3, 0, -N_1 \rangle$ are found and their linear combination $\mathbf{F} = \alpha_1 \mathbf{L} + \alpha_2 \mathbf{L}_\perp + \alpha_3 \mathbf{M} + \alpha_4 \mathbf{M}_\perp + \alpha_5 \mathbf{N} + \alpha_6 \mathbf{N}_\perp$ is also an eigenfunction.

1. Introduction

While teaching a course in vector analysis a question arose about how can you characterize vector fields $\mathbf{F}$ with $\nabla \times \mathbf{F}$ parallel to $\mathbf{F}$. A second question concerned finding interesting examples of $\mathbf{F}$ with $\nabla \times \mathbf{F}$ orthogonal to $\mathbf{F}$. This was in connection with an illustration of differences in behavior between $\mathbf{G} \times \mathbf{F}$, the cross product of a vector $\mathbf{G}$ with $\mathbf{F}$ and $\nabla \times \mathbf{F}$ the curl of $\mathbf{F}$. In fluid dynamics vector fields with $\nabla \times \mathbf{F}$ parallel to $\mathbf{F}$ are called Beltrami fields. Such fields satisfy $\nabla \times \mathbf{F} = \phi \mathbf{F}$ for a scalar function ϕ. In the case when ϕ equals a constant λ or $\nabla \times \mathbf{F} = \lambda \mathbf{F}$ we have a Trklian -Viktor Trkal (1888-1956), Czech physicist and mathematician who worked in theoretical quantum physics- field where λ is an eigenvalue and $\mathbf{F}$ is an eigenfunction of the curl operator. Such fields occur in electromagnetic wave propagation ([4], [6]).

[0]Received August 17, 2006.
[0]Mathematics Subject Classification(2000): 30A99, 26B12, 30A99.
[0]Keywords: Cauchy-Riemann equations, analytic functions, curl operator, Helmholtz equation

Suppose that $\nabla \times \mathbf{F} = \lambda \mathbf{F}$ where λ is a constant. In case $\lambda = 0$, we have $\nabla \times \mathbf{F} = \mathbf{0}$ and $\mathbf{F}$ is conservative. Then if $\mathbf{F}$ has continuous partial derivatives in an open connected set there exists a scalar potential function ϕ such that $\mathbf{F} = \nabla\phi$. In the remaining non-zero case, $\lambda \neq 0$, we have div curl of $\mathbf{F} = \lambda \operatorname{div} \mathbf{F} = \lambda \nabla \cdot \mathbf{F} = 0$, so $\nabla \cdot \mathbf{F} = 0$ and F must be solenoidal. Moreover,

$$\nabla \times (\nabla \times \mathbf{F}) = \lambda \nabla \times \mathbf{F} = \lambda^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 F\,.$$

But $\nabla \cdot \mathbf{F} = 0$ since $\mathbf{F}$ is solenoidal so $\nabla^2 \mathbf{F} + \lambda^2 \mathbf{F} = \mathbf{0}$. Thus $\mathbf{F}$ is a vector valued solution of Helmholtz's equation. Helmholtz's equation comes from the wave equation $(1/c^2)u_{tt} = \nabla^2 u$. If we separate variables via $u(x, y, z, t) = \cos\omega(t - t_0)v(x, y, z)$ then

$$(1/c^2)u_{tt} = -(\omega/c)^2 \cos\omega(t - t_0)v = \cos\omega(t - t_0)\nabla^2 v\,.$$

So it follows that $\nabla^2 v = -(\omega/c)^2 v$ or with $\lambda = \omega/c$,

$$\nabla^2 v + \lambda^2 v = 0 \text{ (Helmholtz's equation)}.$$

In this article vector potential functions for solenoidal fields $\mathbf{F}$ will be found and Trklian fields $\mathbf{F}$ with $\lambda \neq 0$ and one component zero will be completely determined for open convex regions. This will involve a arbitrary analytic function of a complex variable. Finally, for $\lambda \neq 0$, six linearly independent solutions $\mathbf{L} = \langle L_1, L_2, 0\rangle$, $\mathbf{L}_\perp = \langle -L_2, L_1, 0\rangle$, $\mathbf{M} = \langle 0, M_2, M_3\rangle$, $\mathbf{M}_\perp = \langle 0, -M_3, M_2\rangle$, $\mathbf{N} = \langle N_1, 0, N_3\rangle$, and $\mathbf{N}_\perp = \langle N_3, 0, -N_1\rangle$, are found so that $\mathbf{F} = \alpha_1\mathbf{L} + \alpha_2\mathbf{L}_\perp + \alpha_3\mathbf{M} + \alpha_4\mathbf{M}_\perp + \alpha_5\mathbf{N} + \alpha_6\mathbf{N}_\perp$ is a Trklian field where α_i, $i = 1, \ldots, 6$, are arbitrary real constants.

2. Vector potentials for solenoidal vector fields

Let $\mathbf{F} = \langle L, M, N\rangle$ and suppose that $\nabla \cdot \mathbf{F} = L_x + M_y + N_z = 0$ on an open connected set where $\mathbf{F}$ has continuous partial derivatives. Let us first find a vector potential $\mathbf{\Pi} = \langle \Pi_1, \Pi_2, 0\rangle$ with a zero third component. Take the curl of $\mathbf{\Pi}$ to obtain

$$\nabla \times \mathbf{\Pi} = \left\langle -\frac{\partial \Pi_2}{\partial z}, \frac{\partial \Pi_1}{\partial z}, \frac{\partial \Pi_2}{\partial x} - \frac{\partial \Pi_1}{\partial y} \right\rangle$$

so if we have $\nabla \times \mathbf{\Pi} = \mathbf{F}$ we should have

$$L = -\frac{\partial \Pi_2}{\partial z},\ M = \frac{\partial \Pi_1}{\partial z},\ \text{and } N = \frac{\partial \Pi_2}{\partial x} - \frac{\partial \Pi_1}{\partial y}.$$

This leads us to a solution

$$\mathbf{\Pi} = \left\langle \int_{z_0}^{z} M(x, y, t)dt, -\int_{z_0}^{z} L(x, y, t)dt + \int_{x_0}^{x} N(s, y, z_0)ds, 0 \right\rangle.$$

Clearly $-\frac{\partial \Pi_2}{\partial z} = L$, $\frac{\partial \Pi_1}{\partial z} = M$, and

$$\begin{aligned}\frac{\partial \Pi_2}{\partial x} - \frac{\partial \Pi_1}{\partial y} &= -\int_{z_0}^{z} L_x(x,y,t)dt + N(x,y,z_0) - \int_{z_0}^{z} M_y(x,y,t)dt \\ &= \int_{z_0}^{z} N_z(x,y,t)dt + N(x,y,z_0) \\ &= N(x,y,z) - N(x,y,z_0) + N(x,y,z_0) = N\,.\end{aligned}$$

Thus, $\nabla \times \mathbf{\Pi} = \mathbf{F}$. If $R = R(x,y,z)$ is an arbitrary potential then $\mathbf{\Pi} + \nabla R = \langle \Pi_1 + R_x, \Pi_2 + R_y, R_z \rangle$ is the most general vector potential. (See [1], p. 216.)

There is nothing special about the second component being zero. If we want the first component to be zero then

$$\mathbf{P} = \left\langle 0, \int_{x_0}^{x} N(t,y,z)dt, -\int_{x_0}^{x} M(t,y,z)dt + \int_{y_0}^{y} L(x_0,s,z)ds \right\rangle$$

is a vector potential for $\mathbf{F}$ and if we want the 2^{nd} component to be zero then

$$\mathbf{Q} = \left\langle -\int_{y_0}^{y} N(x,t,z)dt + \int_{z_0}^{z} M(x,y_0,s)ds, 0, \int_{y_0}^{y} L(x,t,z)dt \right\rangle$$

is also a vector potential for $\mathbf{F}$.

Since Trklian vector fields must be solenoidal and since solenoidal vector fields have vector potentials with one zero component it is natural to try to find all Trklian vector fields with one zero component.

3. Trklian vector fields with one zero component and $\lambda \neq 0$.

Let $\mathbf{L} = \langle L_1, L_2, 0 \rangle$ be a Trklian vector field with a zero third component and with twice continuous partial derivatives on an open convex set. Then

$$\nabla \times \mathbf{L} = \left\langle -\frac{\partial L_2}{\partial z}, \frac{\partial L_1}{\partial z}, \frac{\partial L_2}{\partial z} - \frac{\partial L_1}{\partial z} \right\rangle = \langle \lambda L_1, \lambda L_2, 0 \rangle = \lambda \mathbf{L}$$

and

$$\nabla \cdot \mathbf{L} = 0$$

gives us

$$\frac{\partial L_1}{\partial z} = \lambda L_2 \tag{1}$$

$$\frac{\partial L_2}{\partial z} = -\lambda L_1 \tag{2}$$

$$\frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial y} = 0 \tag{3}$$

$$\frac{\partial L_1}{\partial x} + \frac{\partial L_2}{\partial y} = 0\,. \tag{4}$$

In (1) take another partial derivative with respect to z to obtain

$$\frac{\partial^2 L_1}{\partial z^2} = -\lambda^2 L_1 \text{ or } \frac{\partial^2 L_1}{\partial z^2} + \lambda^2 L_1 = 0$$

and in (2) take another partial derivative with respect to z to obtain

$$\frac{\partial^2 L_2}{\partial z^2} = -\lambda^2 L_2 \text{ or } \frac{\partial^2 L_2}{\partial z^2} + \lambda^2 L_2 = 0\,.$$

The general solutions of these equations are given by

$$L_1 = v(x,y)\cos(\lambda(z-z_0)) + u(x,y)\sin(\lambda(z-z_0))$$

and

$$L_2 = A(x,y)\cos(\lambda(z-z_0)) + B(x,y)\sin(\lambda(z-z_0))\ .$$

Now from (1) we have

$$\begin{aligned}\frac{\partial L_1}{\partial z} &= \lambda u(x,y)\cos(\lambda(z-z_0)) - \lambda r(x,y)\sin(\lambda(z-z_0)) \\ &= \lambda A(x,y)\cos(\lambda(z-z_0)) + \lambda B(x,y)\sin(\lambda(z-z_0))\end{aligned}$$

and by the linear independence of $\cos(\lambda(z-z_0))$ and $\sin(\lambda(z-z_0))$ it follows that $A = u$ and $B = -v$. Thus,

$$L_1 = v\cos(\lambda(z-z_0)) + u\sin(\lambda(z-z_0))$$

and

$$L_2 = u\cos(\lambda(z-z_0)) - v\sin(\lambda(z-z_0))\,.$$

From (2) we have $\frac{\partial L_2}{\partial z} = -\lambda v\cos(\lambda(z-z_0)) - \lambda u\sin(\lambda(z-z_0)) = -\lambda L_1$ which doesn't give us anything new. Now from (3) we have

$$(u_x - v_y)\cos(\lambda(z-z_0)) + (-v_x - u_y)\sin(\lambda(z-z_0)) = 0.$$

Then the linear independence of $\cos(\lambda(z-z_0))$ and $\sin(\lambda(z-z_0))$ gives us $u_x = v_y$ and $v_x = -u_y$ or $u_x + iv_x = -i(u_y + iv_y) = v_y - iu_y$ which are the Cauchy-Riemann equations for u and v. We have

$$\mathbf{L} = \langle v\cos(\lambda(z-z_0)) + u\sin(\lambda(z-z_0)), u\cos(\lambda(z-z_0)) - v\sin(\lambda(z-z_0)), 0\rangle$$

which has twice continuous partial derivatives on an open convex set. For each value of z the resulting plane of intersection with the open convex set is an open convex set. The Cauchy-Riemann equations hold on this set so $f(x+iy) = u(x,y)+iv(x,y)$ is analytic with $f'(x+iy) = u_x + iv_x = -i(u_y + iv_y) = v_y - iu_y$ (See [2] and [5]). But observe now that $L_2 + iL_1 = (u+iv)\mathrm{e}^{i\lambda(z-z_0)}$ or $\mathrm{Re}((u+iv)\mathrm{e}^{i\lambda(z-z_0)}) = L_2$ and $\mathrm{Im}((u+iv)\mathrm{e}^{i\lambda(z-z_0)}) = L_1$.

The vector field $\mathbf{L}$ is suppose to be solenoidal. Let us check that our solution is solenoidal. We have

$$\frac{\partial L_1}{\partial x} + \frac{\partial L_2}{\partial y} = (v_x + u_y)\cos(\lambda(z-z_0)) + (u_x - v_y)\sin(\lambda(z-z_0)) = 0$$

since $u_x = u_y$ and $v_x = -u_y$ by the Cauchy-Riemann equations for u and v.

Let us check that $\mathbf{L}$ satisfies Helmholtz's equation. Putting our $\mathbf{L}$ into $\mathbf{L}_{xx} + \mathbf{L}_{yy} + \mathbf{L}_{zz} + \lambda^2\mathbf{L}$ this equals

$$\begin{aligned}\langle (v_{xx}+v_{yy})\cos(\lambda(z-z_0)) + (u_{xx}+u_{yy})\sin(\lambda(z-z_0)),\\ (u_{xx}+u_{yy})\cos(\lambda(z-z_0)) - (v_{xx}+v_{yy})\sin(\lambda(z-z_0)), 0\rangle\\ -\lambda^2\mathbf{L} + \lambda^2\mathbf{L} = \mathbf{0}\end{aligned}$$

since both u and v are harmonic functions.

Thus, $\mathbf{L} =$

$$\langle v\cos(\lambda(z-z_0)) + u\sin(\lambda(z-z_0)), u\cos(\lambda(z-z_0)) - v\sin(\lambda(z-z_0)), 0\rangle$$

is the general solution of $\nabla \times \mathbf{L} = \lambda\mathbf{L}$ with $\lambda \neq 0$ and $\mathbf{L} = \langle L_1, L_2, 0\rangle$ where $L_2 + iL_1 = f(x+iy)\mathrm{e}^{i\lambda(z-z_0)} = (u+iv)\mathrm{e}^{i\lambda(z-z_0)}$ and $f(x+iy)$ is an arbitrary analytic function of $x+iy$. Observe that $||\mathbf{L}|| = \sqrt{L_1^2 + L_2^2} = \sqrt{u^2+v^2} = |f|$. In $\mathbf{L}$ if "λ" is replaced with "$-\lambda$" we obtain

$$\langle v\cos(\lambda(z-z_0)) - u\sin(\lambda(z-z_0)), u\cos(\lambda(z-z_0)) + v\sin(\lambda(z-z_0)), 0\rangle$$

subtract $\mathbf{L}$ from this vector and divide by "2" to obtain
$\mathbf{P} = \langle u, -v, 0\rangle \sin(\lambda(z-z_0))$. Add $\mathbf{L}$ and this vector and divide by "2" to obtain $\mathbf{Q} = \langle v, u, 0\rangle\cos(\lambda(z-z_0))$. Now, $\mathbf{P}\cdot\mathbf{Q} = 0$ so $\mathbf{P}$ & $\mathbf{Q}$ are orthogonal. If we take the cross product of $\mathbf{P}$ and $\mathbf{Q}$ then

$$\begin{aligned}\mathbf{P}\times\mathbf{Q} &= (u^2+v^2)\sin(\lambda(z-z_0))\cos(\lambda(z-z_0))\langle 0,0,1\rangle\\ &= |f|^2\sin(\lambda(z-z_0))\cos(\lambda(z-z_0))\langle 0,0,1\rangle\end{aligned}$$

which is orthogonal to the xy-plane. Taking the curl of $\mathbf{P}$ and $\mathbf{Q}$ we obtain $\nabla\times\mathbf{P} = \lambda\mathbf{Q}$ and $\nabla\times\mathbf{Q} = \lambda\mathbf{P}$. We have $\mathbf{L} = \mathbf{P}+\mathbf{Q}$ so $\nabla\times\mathbf{L} = \lambda\mathbf{Q}+\lambda\mathbf{Q} = \lambda\mathbf{L}$ and we also have $\nabla\times(\mathbf{P}-\mathbf{Q}) = \lambda(\mathbf{P}-\mathbf{Q}) = (-\lambda)(\mathbf{P}-\mathbf{Q})$. The vectors $\mathbf{P}$ and $\mathbf{Q}$ are orthogonal to their curls and the vectors $\mathbf{P}\pm\mathbf{Q}$ are parallel to their curls. Moreover, both $\mathbf{P}$ and $\mathbf{Q}$ are solutions of Helmholtz's equation and both are solenoidal.

The vector field $L_\perp = \langle -L_2, L_1, 0\rangle$ is orthogonal to $L = \langle L_1, L_2, 0\rangle$. Taking the curl and divergence of $L_\perp$ we get

$$\begin{aligned}\nabla \times \mathbf{L}_\perp &= \left\langle \frac{\partial L_1}{\partial z}, \frac{\partial L_2}{\partial z}, \frac{\partial L_1}{\partial x} + \frac{\partial L_2}{\partial y} \right\rangle \\ &= \langle -\lambda L_2, \lambda L_1, 0\rangle = \lambda \mathbf{L}_\perp\end{aligned}$$

and

$$\nabla \cdot \mathbf{L}_\perp = -\frac{\partial L_2}{\partial x} + \frac{\partial L_1}{\partial y} = 0$$

because $\mathbf{L}$ satisfies (1), (2), (3), and (4). Thus, $\mathbf{L}$ and $\mathbf{L}_\perp$ are orthogonal eigenfunctions of the curl operator which lie in the xy-plane. $\mathbf{L}_\perp = \langle -L_2, L_1, 0\rangle$ and $L_1 - iL_2 = -i(L_2 + iL_1) = -if(x+iy)\mathrm{e}^{i\lambda(z-z_0)} = -(u+iv)\mathrm{e}^{i\lambda(z-z_0)}$ where $f(x+iy)$ is the arbitrary analytic function of $x+iy$ corresponding to $\mathbf{L}$. $f(x+iy)$ and $-if(x+iy)$ are orthogonal to each other in the complex plane and $|f(x+iy)| = |-if(x+iy)| = |f| = \|\mathbf{L}\| = \|\mathbf{L}_\perp\|$. For $\mathbf{L}_\perp$ the corresponding $\mathbf{P}_\perp$ and $\mathbf{Q}_\perp$ are $\mathbf{P}_\perp = \langle v, u, 0\rangle \sin\lambda(z-z_0)$ and $\mathbf{Q}_\perp = \langle -u, v, 0\rangle \cos\lambda(z-z_0)$.

Then $\mathbf{L}_\perp = \mathbf{P}_\perp + \mathbf{Q}_\perp$, $\nabla \times \mathbf{P}_\perp = \lambda \mathbf{Q}_\perp$, $\nabla \times \mathbf{Q}_\perp = \lambda \mathbf{P}_\perp$, $\nabla \times (\mathbf{P}_\perp \pm \mathbf{Q}_\perp) = \pm(\mathbf{P}_\perp \pm \mathbf{Q}_\perp)$, and

$$\mathbf{P}_\perp \times \mathbf{Q}_\perp = |f|^2 \sin\lambda(z-z_0)\cos\lambda(z-z_0)\langle 0,0,1\rangle\,.$$

Moreover, both $\mathbf{P}_\perp$ and $\mathbf{Q}_\perp$ are solutions of Helmholtz's equation and solenoidal.

There is nothing special about the third component being zero. If $\mathbf{M} = \langle 0, M_2, M_3\rangle$ is a Trklian vector field with a zero first component and with twice continuous partial derivatives on an open convex set then we have

$$M_3 + iM_2 = g(y+iz)\mathrm{e}^{i\lambda(x-x_0)} = (l+im)\mathrm{e}^{i\lambda(x-x_0)}$$

where $g(y+iz) = l(y,z) + im(y,z)$ is an analytic function of $y+iz$ and

$$\mathbf{M} = \langle 0, m\cos(\lambda(x-x_0)) + l\sin(\lambda(x-x_0)),\, l\cos(\lambda(x-x_0)) - m\sin(\lambda(x-x_0))\rangle$$

is the general solution of $\nabla \times \mathbf{M} = \lambda\mathbf{M}$ with $\lambda \neq 0$ and $\mathbf{M} = \langle 0, M_2, M_3\rangle$. Here we have $\|\mathbf{M}\| = \sqrt{m^2+l^2} = |g|$. In this case if

$$\begin{aligned}\mathbf{R} &= \langle 0, l, -m\rangle \sin(\lambda(x-x_0)) \quad \text{and} \\ \mathbf{S} &= \langle 0, m, l\rangle \cos(\lambda(x-x_0))\end{aligned}$$

then $\mathbf{R}\cdot\mathbf{S} = 0$ so $\mathbf{R}$ and $\mathbf{S}$ are orthogonal and

$$\begin{aligned}\mathbf{R}\times\mathbf{S} &= (l^2+m^2)\sin(\lambda(x-x_0))\cos(\lambda(x-x_0))\langle 1,0,0\rangle \\ &= |g|^2 \sin(\lambda(x-x_0))\cos(\lambda(x-x_0))\langle 1,0,0\rangle\end{aligned}$$

which is orthogonal to the yz-plane. Again we have

$$\begin{aligned}\nabla \times \mathbf{R} &= \lambda \mathbf{S}\\ \nabla \times \mathbf{S} &= \lambda \mathbf{R}\\ \nabla \times \mathbf{M} = \nabla \times (\mathbf{R}+\mathbf{S}) &= \lambda \mathbf{M} \quad \text{and}\\ \nabla \times (\mathbf{R}-\mathbf{S}) &= (-\lambda)(\mathbf{R}-\mathbf{S})\,.\end{aligned}$$

Thus, $\mathbf{P}$ and $\mathbf{Q}$ are orthogonal to their curls and the vectors $\mathbf{P} \pm \mathbf{Q}$ are parallel to their curls. Again both $\mathbf{P}$ and $\mathbf{Q}$ are solutions of Helmholz's equation and both are solenoidal.

The vector field $\mathbf{M}_\perp = \langle 0, -M_3, M_2\rangle$ is orthogonal to $\mathbf{M}$ and is another eigenfunction of the curl operator which lies in the yz-plane. We have $M_2 - iM_3 = -i(M_3 + iM_2) = -ig(y+iz)\mathrm{e}^{i\lambda(x-x_0)} = -i(\ell + im)\mathrm{e}^{i\lambda(x-x_0)}$ where $g(y+iz)$ is the analytic function corresponding to M. As above we have $|g| = \|\mathbf{M}\| = \|\mathbf{M}_\perp\|$. For $\mathbf{M}_\perp$ the corresponding $\mathbf{R}_\perp$ and $\mathbf{S}_\perp$ are $\mathbf{R}_\perp = \langle 0, m, \ell\rangle \sin\lambda(x-x_0)$ and $\mathbf{S}_\perp = \langle 0, -\ell, m\rangle \cos\lambda(x-x_0)$. Then $\mathbf{M}_\perp = \mathbf{R}_\perp + \mathbf{S}_\perp$, $\nabla \times \mathbf{R}_\perp = \lambda \mathbf{S}_\perp$, $\nabla \times \mathbf{S}_\perp = \lambda \mathbf{R}_\perp$,

$$\begin{aligned}\nabla \times (\mathbf{R}_\perp \pm \mathbf{S}_\perp) &= \pm\lambda(\mathbf{R}_\perp + \mathbf{S}_\perp), \text{ and}\\ \mathbf{R}_\perp \times \mathbf{S}_\perp &= |q|^2 \sin\lambda(x-x_0)\lambda(x-x_0)\langle 1,0,0\rangle\,.\end{aligned}$$

Moreover, both $\mathbf{R}_\perp$ and $\mathbf{S}_\perp$ are solutions of Helmholtz's equation and are solenoidal.

If $\mathbf{N} = \langle N_1, 0, M_3\rangle$ is a Trklian vector field with a zero second component and with twice continuous partial derivatives on an open convex set then we have

$$N_1 + iN_3 = h(z+ix)\mathrm{e}^{i\lambda(y-y_0)} = (r+is)\mathrm{e}^{i\lambda(y-y_0)}$$

where $h(z+ix) = r(z,x) + is(z,x)$ is an analytic function of $z+ix$ and

$$\mathbf{N} = \langle r\cos(\lambda(y-y_0)) - s\sin(\lambda(y-y_0)), 0, s\cos(\lambda(y-y_0)) + r\sin(\lambda(y-y_0))\rangle$$

is the general solution of $\nabla \times \mathbf{N} = \lambda\mathbf{N}$ with $\lambda \neq 0$ and $\mathbf{N} = \langle N_1, 0, N_3\rangle$. Here we have $\|\mathbf{N}\| = \sqrt{r^2+s^2} = |h|$. In this case if

$$\mathbf{T} = \langle -s, 0, r\rangle \sin(\lambda(y-y_0))$$

and

$$\mathbf{W} = \langle r, 0, s\rangle \cos(\lambda(y-y_0))$$

then $\mathbf{T}\cdot\mathbf{W} = 0$ so $\mathbf{T}$ and $\mathbf{W}$ are orthogonal and

$$\begin{aligned}\mathbf{T} \times \mathbf{W} &= (r^2+s^2)\sin(\lambda(y-y_0))\cos(\lambda(y-y_0))\\ &= |h|^2 \sin(\lambda(y-y_0))\cos(\lambda(y-y_0))\langle 0,1,0\rangle\end{aligned}$$

which is orthogonal to the zx-plane. Again we have $\nabla \times \mathbf{T} = \lambda\mathbf{W}$, $\nabla \times \mathbf{W} = \lambda\mathbf{T}$, $\nabla \times \mathbf{N} = \nabla \times (\mathbf{T}+\mathbf{W}) = \lambda\mathbf{N}$, and $\nabla \times (\mathbf{T}-\mathbf{W}) = -\lambda(\mathbf{T}-\mathbf{W})$. Thus, $\mathbf{T}$ and $\mathbf{W}$ are orthogonal to their curls and the vectors $\mathbf{T} \pm \mathbf{W}$ are parallel to their curls. Again both $\mathbf{T}$ and $\mathbf{L}$ are solutions of Helmholtz's equation and both are solenoidal.

The vector field $\mathbf{N}_\perp = \langle N_3, 0, -N_1\rangle$ is orthogonal to $\mathbf{N}$ and is another eigenfunction of the curl operator which lies in the zx-plane. We have

$$N_3 - iN_1 = -i(N_1 + iM_3) = -ih(z+ix)\mathrm{e}^{i\lambda(y-y_0)} = -i(r+is)\mathrm{e}^{\lambda(y-y_0)}$$

where $h(z+ix)$ is the analytic function corresponding to $\mathbf{N}$. As above we have $|h| = \|\mathbf{N}\| = \|\mathbf{N}_\perp\|$. For $\mathbf{N}_\perp$ the corresponding $\mathbf{T}_\perp$ and $\mathbf{W}_\perp$ are $\mathbf{T}_\perp = \langle r, 0, s\rangle \sin\lambda(y-y_0)$ and $\mathbf{W}_\perp = \langle s, 0, -r\rangle \cos\lambda(y-y_0)$. Then $\mathbf{N}_\perp = \mathbf{T}_\perp + \mathbf{W}_\perp$, $\nabla\times\mathbf{T}_\perp = \lambda\mathbf{W}_\perp$, $\nabla\times\mathbf{W}_\perp = \lambda\mathbf{T}_\perp$, $\nabla\times(\mathbf{T}_\perp \pm \mathbf{W}_\perp) = (\pm\lambda)(\mathbf{T}_\perp \pm \mathbf{W}_\perp)$, and

$$\mathbf{T}_\perp \times \mathbf{W}_\perp = |h|^2 \sin\lambda(y-y_0)\cos\lambda(y-y_0)\langle 0,1,0\rangle\,.$$

Moreover, both $\mathbf{T}_\perp$ and $\mathbf{W}_\perp$ are solutions of Helmholtz's equation and are solenoidal.

4. The linear independence of the vectors $\mathbf{L}$, $\mathbf{L}_\perp$, $\mathbf{M}$, $\mathbf{M}_\perp$, $\mathbf{N}$, & $\mathbf{N}_\perp$ If $\mathbf{f}'$, $\mathbf{g}'$, and $\mathbf{h}'$ are non-zero and continuous.

If $\phi(\xi+i\eta)$ is analytic in an open connected set D and if $\phi'(\xi+i\eta)$ is non-zero and continuous on D then the mapping $\xi+i\eta \to \phi(\xi+i\eta)$ is locally one-to-one (See [3], Theorem 4.6.1, p. 91).

Proposition. *Suppose that the functions $f(x+iy)$, $g(y+iz)$, and $h(z+ix)$ are analytic and have continuous non-zero derivatives on an open convex set. Then the corresponding $\mathbf{L} = \langle L_1, L_2, 0\rangle$, $\mathbf{L}_\perp = \langle -L_2, L_1, 0\rangle$, $\mathbf{M} = \langle 0, M_2, M_3\rangle$, $\mathbf{M}_\perp = \langle 0, -M_3, M_2\rangle$, $\mathbf{N} = \langle N_1, 0, N_3\rangle$, and $\mathbf{N}_\perp = \langle N_3, 0, -N_1\rangle$ with*

$$\begin{aligned}
L_2 + iL_1 &= f(x+iy)\mathrm{e}^{i\lambda(z-z_0)} = (u+iv)\mathrm{e}^{i\lambda(z-z_0)},\\
L_1 - iL_2 &= -if(x+iy)\mathrm{e}^{i\lambda(z-z_0)} = -i(u+iv)\mathrm{e}^{i\lambda(z-z_0)},\\
M_3 + iM_2 &= g(y+iz)\mathrm{e}^{i\lambda(x-x_0)} = (\ell+im)\mathrm{e}^{i\lambda(x-x_0)},\\
M_2 - iM_3 &= -ig(y+iz)\mathrm{e}^{i\lambda(x-x_0)} = -i(\ell+im)\mathrm{e}^{i\lambda(x-x_0)},\\
N_1 + iN_3 &= h(z+ix)\mathrm{e}^{i\lambda(y-y_0)} = (r+is)\mathrm{e}^{i\lambda(y-y_0)},
\end{aligned}$$

and

$$N_3 - iN_1 = -ih(z+ix)\mathrm{e}^{i\lambda(y-y_0)} = -i(r+is)\mathrm{e}^{i\lambda(y-y_0)}$$

are linearly independent if $\lambda \neq 0$.

Proof. Set

$$\alpha_1\mathbf{L} + \alpha_2\mathbf{L}_\perp + \alpha_3\mathbf{M} + \alpha_4\mathbf{M}_\perp + \alpha_5\mathbf{N} + \alpha_6\mathbf{N}_\perp = 0,$$

where α_i, $i = 1, \ldots, 6$, are real constants. Then we have the following equations:

$$\alpha_1 L_1 - \alpha_2 L_2 + \alpha_5 N_1 + \alpha_6 N_3 = 0,$$
$$\alpha_1 L_2 + \alpha_2 L_1 + \alpha_3 M_2 - \alpha_4 M_3 = 0,$$

and

$$\alpha_3 M_3 + \alpha_4 M_2 + \alpha_5 N_3 - \alpha_6 N_1 = 0\,.$$

Take the following partial derivatives:

$$\alpha_1 \frac{\partial^2 L_1}{\partial y \partial z} - \alpha_2 \frac{\partial^2 L_2}{\partial y \partial z} + \alpha_5 \frac{\partial^2 N_1}{\partial y \partial z} + \alpha_6 \frac{\partial^2 N_3}{\partial y \partial z} = 0,$$
$$\alpha_1 \frac{\partial^2 L_2}{\partial z \partial x} + \alpha_2 \frac{\partial^2 L_1}{\partial z \partial x} + \alpha_3 \frac{\partial^2 M_2}{\partial z \partial x} - \alpha_4 \frac{\partial^2 M_3}{\partial z \partial x} = 0,$$

and

$$\alpha_3 \frac{\partial^2 M_3}{\partial x \partial y} + \alpha_4 \frac{\partial^2 M_2}{\partial x \partial y} + \alpha_5 \frac{\partial^2 N_3}{\partial x \partial y} - \alpha_6 \frac{\partial^2 N_1}{\partial x \partial y} = 0\,.$$

We start with

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} &= \alpha_1 \begin{pmatrix} v\cos(\lambda(z-z_0)) + u\sin(\lambda(z-z_0)) \\ u\cos(\lambda(z-z_0)) - v\sin(\lambda(z-z_0)) \\ 0 \end{pmatrix} \\
&+ \alpha_2 \begin{pmatrix} -u\cos(\lambda(z-z_0)) + v\sin(\lambda(z-z_0)) \\ v\cos(\lambda(z-z_0)) + u\sin(\lambda(z-z_0)) \\ 0 \end{pmatrix} \\
&+ \alpha_3 \begin{pmatrix} 0 \\ m\cos(\lambda(x-x_0)) + \ell\sin(\lambda(x-x_0)) \\ \ell\cos(\lambda(x-x_0)) - m\sin(\lambda(x-x_0)) \end{pmatrix} \\
&+ \alpha_4 \begin{pmatrix} 0 \\ -\ell\cos(\lambda(x-x_0)) + m\sin(\lambda(x-x_0)) \\ m\cos(\lambda(x-x_0)) + \ell\sin(\lambda(x-x_0)) \end{pmatrix} \\
&+ \alpha_5 \begin{pmatrix} r\cos(\lambda(y-y_0)) - s\sin(\lambda(y-y_0)) \\ 0 \\ s\cos(\lambda(y-y_0)) + r\sin(\lambda(y-y_0)) \end{pmatrix} \\
&+ \alpha_6 \begin{pmatrix} s\cos(\lambda(y-y_0)) + r\sin(\lambda(y-y_0)) \\ 0 \\ -r\cos(\lambda(y-y_0)) + s\sin(\lambda(y-y_0)) \end{pmatrix},
\end{aligned}$$

and obtain

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \alpha_1 \begin{pmatrix}-\lambda v_y \sin(\lambda(z-z_0)) + \lambda u_y \cos(\lambda(z-z_0))\\ -\lambda u_x \sin(\lambda(z-z_0)) - \lambda v_x \cos(\lambda(z-z_0))\\ 0\end{pmatrix}$$
$$+ \alpha_2 \begin{pmatrix}\lambda u_y \sin(\lambda(z-z_0)) + \lambda v_y \cos(\lambda(z-z_0))\\ -\lambda v_x \sin(\lambda(z-z_0)) + \lambda u_x \cos(\lambda(z-z_0))\\ 0\end{pmatrix}$$
$$+ \alpha_3 \begin{pmatrix}0\\ -\lambda m_z \sin(\lambda(x-x_0)) + \lambda \ell_z \cos(\lambda(x-x_0))\\ -\lambda \ell_y \sin(\lambda(x-x_0)) - \lambda m_y \cos(\lambda(x-x_0))\end{pmatrix}$$
$$+ \alpha_4 \begin{pmatrix}0\\ \lambda \ell_z \sin(\lambda(x-x_0)) + \lambda m_z \cos(\lambda(x-x_0))\\ -\lambda m_y \sin(\lambda(x-x_0)) + \lambda \ell_y \cos(\lambda(x-x_0))\end{pmatrix}$$
$$+ \alpha_5 \begin{pmatrix}-\lambda r_z \sin(\lambda(y-y_0)) - \lambda s_z \cos(\lambda(y-y_0))\\ 0\\ -\lambda s_x \sin(\lambda(y-y_0)) + \lambda r_x \cos(\lambda(y-y_0))\end{pmatrix}$$
$$+ \alpha_6 \begin{pmatrix}-\lambda s_z \sin(\lambda(y-y_0)) + \lambda r_z \cos(\lambda(y-y_0))\\ 0\\ \lambda r_x \sin(\lambda(y-y_0)) - \lambda s_x \cos(\lambda(y-y_0))\end{pmatrix}.$$

From the Cauchy-Riemann equations we have

$$f'(x+iy) = u_x + iv_x = -i(u_y + iv_y) = v_y - iu_y,$$
$$g'(y+iz) = \ell_y + im_y = -i(\ell_z + im_z) = m_z - i\ell_z,$$

and

$$h'(z+ix) = r_z + is_z = -i(r_x + is_x) = s_x - ir_x\,.$$

That gives us

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \alpha_1 \begin{pmatrix}-\lambda u_x \sin(\lambda(z-z_0)) - \lambda v_x \cos(\lambda(z-z_0))\\ -\lambda u_x \sin(\lambda(z-z_0)) - \lambda v_x \cos(\lambda(z-z_0))\\ 0\end{pmatrix}$$
$$+ \alpha_2 \begin{pmatrix}-\lambda v_x \sin(\lambda(z-z_0)) + \lambda u_x \cos(\lambda(z-z_0))\\ -\lambda v_x \sin(\lambda(z-z_0)) + \lambda u_x \cos(\lambda(z-z_0))\\ 0\end{pmatrix}$$

$$
+ \alpha_3 \begin{pmatrix} 0 \\ -\lambda \ell_y \sin(\lambda(x-x_0)) - \lambda m_y \cos(\lambda(x-x_0)) \\ -\lambda \ell_y \sin(\lambda(x-x_0)) - \lambda m_y \cos(\lambda(x-x_0)) \end{pmatrix}
$$
$$
+ \alpha_4 \begin{pmatrix} 0 \\ -\lambda m_y \sin(\lambda(x-x_0)) + \lambda \ell_y \cos(\lambda(x-x_0)) \\ -\lambda m_y \sin(\lambda(x-x_0)) + \lambda \ell_y \cos(\lambda(x-x_0)) \end{pmatrix}
$$
$$
+ \alpha_5 \begin{pmatrix} -\lambda r_z \sin(\lambda(y-y_0)) - \lambda s_z \cos(\lambda(y-y_0)) \\ 0 \\ -\lambda r_z \sin(\lambda(y-y_0)) - \lambda s_z \cos(\lambda(y-y_0)) \end{pmatrix}
$$
$$
+ \alpha_6 \begin{pmatrix} -\lambda s_z \sin(\lambda(y-y_0)) + \lambda r_z \cos(\lambda(y-y_0)) \\ 0 \\ -\lambda s_z \sin(\lambda(y-y_0)) + \lambda r_z \cos(\lambda(y-y_0)) \end{pmatrix},
$$

which can be rewritten as

$$
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = -\lambda \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}
\times \begin{pmatrix} (\alpha_1 u_x + \alpha_2 v_x) \sin\lambda(z-z_0) + (\alpha_1 v_x - \alpha_2 u_x) \cos\lambda(z-z_0) \\ (\alpha_3 \ell_y + \alpha_4 m_y) \sin\lambda(x-x_0) + (\alpha_3 m_y - \alpha_4 \ell_y) \cos\lambda(x-x_0) \\ (\alpha_5 r_z + \alpha_6 s_z) \sin\lambda(y-y_0) + (\alpha_5 s_z - \alpha_6 r_z) \cos\lambda(y-y_0) \end{pmatrix}.
$$

The matrix

$$
\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}
$$

has a determinant 2 so it is invertible. The scalar λ is non-zero and hence we must have

$$
(\alpha_1 u_x + \alpha_2 v_x) \sin\lambda(z-z_0) + (\alpha_1 v_x - \alpha_2 u_x) \cos\lambda(z-z_0) = 0,
$$
$$
(\alpha_1 \ell_y + \alpha_2 m_y) \sin\lambda(x-x_0) + (\alpha_1 m_y - \alpha_2 \ell_y) \cos\lambda(x-x_0) = 0,
$$

and

$$
(\alpha_1 r_z + \alpha_2 s_z) \sin\lambda(y-y_0) + (\alpha_1 s_z - \alpha_2 r_z) \cos\lambda(y-y_0) = 0\,.
$$

Since the sine and cosine functions are linearly independent we must have the following:

$$
\begin{pmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_1 \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$
$$
\begin{pmatrix} \alpha_3 & \alpha_4 \\ -\alpha_4 & \alpha_3 \end{pmatrix} \begin{pmatrix} \ell_y \\ m_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

and

$$\begin{pmatrix} \alpha_5 & \alpha_6 \\ -\alpha_6 & \alpha_5 \end{pmatrix} \begin{pmatrix} r_z \\ s_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Multiplying each of these coefficient matrices by its adjoint we obtain the following results:

$$(\alpha_1^2 + \alpha_2^2) \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$(\alpha_3^2 + \alpha_4^2) \begin{pmatrix} \ell_y \\ m_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$(\alpha_5^2 + \alpha_6^2) \begin{pmatrix} r_z \\ s_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

But this is equivalent to having

$$(\alpha_1^2 + \alpha_2^2) f'(x+iy) = 0,$$
$$(\alpha_3^2 + \alpha_4^2) g'(x+iy) = 0,$$

and

$$(\alpha_5^2 + \alpha_6^2) h'(x+iy) = 0.$$

Since the derivatives are non-zero on an open convex set we must have all the constants equal to zero. Thus the functions $\mathbf{L}, \mathbf{L}_\perp, \mathbf{M}, \mathbf{M}_\perp, \mathbf{N}$, and $\mathbf{N}_\perp$ are linearly independent. $\square$

Now if $\mathbf{F} = \alpha_1\mathbf{L} + \alpha_2\mathbf{L}_\perp + \alpha_3\mathbf{M} + \alpha_4\mathbf{M}_\perp + \alpha_5\mathbf{N} + \alpha_6\mathbf{N}_\perp$ where α_i, $i = 1, \ldots, 6$, are arbitrary constants then $\nabla \times \mathbf{F} = \lambda F$. So all vectors in span$\{\mathbf{L}, \mathbf{L}_\perp, \mathbf{M}, \mathbf{M}_\perp, \mathbf{N}, \mathbf{N}_\perp\}$ are eigenvectors of the curl operator.

Remark. If α_1 and α_2 are arbitrary constants then $\alpha_1\mathbf{L} + \alpha_2\mathbf{L}_\perp = \langle \alpha_1 L_1 - \alpha_2 L_2, \alpha_1 L_2 + \alpha_2 L_1, 0\rangle$ and

$$\begin{aligned}
(\alpha_1 L_2 + \alpha_2 L_1) &+ i(\alpha_1 L_1 - \alpha_2 L_2) \\
&= \alpha_1(L_2 + iL_2) + \alpha_2(L_1 - iL_2) \\
&= \alpha_1(u+iv)\mathrm{e}^{i\lambda(z-z_0)} - i\alpha_2(u+iv)\mathrm{e}^{i\lambda(z-z_0)} \\
&= (\alpha_1 - i\alpha_2)(u+iv)\mathrm{e}^{i\lambda(z-z_0)} \\
&= (\alpha_1 - i\alpha_2)(u+iv)\mathrm{e}^{i\lambda(z-z_0)} \\
&= (\alpha_1 - i\alpha_2) f(x+iy)\mathrm{e}^{i\lambda(z-z_0)}
\end{aligned}$$

where $L_2 + iL_1 = f(x+iy)\mathrm{e}^{i\lambda(z-z_0)}$. Thus if $\mathbf{L}$ corresponds to $f(x+iy)$ then $\alpha_1\mathbf{L} + \alpha_2\mathbf{L}_\perp$ corresponds to $(\alpha_1 - i\alpha_2) f(x+iy)\mathrm{e}^{i\lambda(z-z_0)}$.

Likewise $\alpha_2\mathbf{M} + \alpha_4\mathbf{M}_\perp$ corresponds to $(\alpha_3 - i\alpha_4)g(y+iz)\mathrm{e}^{i\lambda(x-x_0)}$ where $\mathbf{M}$ corresponds to $g(y+iz)\mathrm{e}^{i\lambda(x-x_0)}$ and $\alpha_5\mathbf{N} + \alpha_6\mathbf{N}_\perp$ corresponds to $(\alpha_5 - i\alpha_6)h(z+ix)\mathrm{e}^{i\lambda(y-y_0)}$ where $\mathbf{N}$ corresponds to $h(z+ix)\mathrm{e}^{i\lambda(y-y_0)}$.

References

[1] Borisenko, A. I. and Tarapov, I. E., *Vector and Tensor Analysis with Applications*, Dover Publications Inc., 1979.

[2] Brown, J. W. and Churchill, R. V., *Complex Variables and Applications*, 7th Ed., McGraw Hill, 2004.

[3] Hille, E., *Analytic Function Theory*, Vol. I, Ginn and Company, 1959.

[4] Lakhtakia, A., *Viktor Trkal, Beltrami fields, and Trkalian flows*, Czechslovak Journal of Physics, Springer Netherlands, Vol. 44, No. 2, Feb. 1994, 89-96. (An introduction to an English translation of an article by V. Trkal.)

[5] Saff, E. B. and Snider, A. D., *Fundamentals of Complex Analysis*, 3rd Ed., Pearson Education Inc., 2003.

[6] Trkal, V., *A note on the hydrodynamics of viscous fluids*, Czechoslovak Journal of Physics, Springer Netherlands, Vol. 44, No 2, Feb. 1994, 97-106. (An English translation of an article by V. Trkal from 1919.)