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APPROXIMATION AND STABILITY RESULTS FOR $S(a,b,c)$-CONTRACTION MAPPINGS

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Abstract. The purpose of this paper is to prove the existence of fixed point for a mapping T define on closed subset of Banach space X which belongs to the class $S(a,b,c)$-contraction, where a,b,c are nonnegative real numbers such that $a+b+c<1$. Also, we deal with the problem of approximation of fixed point and its stability via generalized Ishikawa iteration process of rank 3 introduced by Sahu [36] in a Banach space. Our result generalizes several results in this direction.

1. Introduction

Let C be a nonempty subset of a metric space (X,d) and T be a mapping from C into itself. Then T is said to be

(i) contraction [1] if there exists a number $k \in (0,1)$ such that

$$d(Tx,Ty) \le kd(x,y) \tag{1.1}$$

for all $x,y \in C$.

(ii) Kannan type mapping [19] if there exists a number $k \in (0,\frac{1}{2})$ such that

$$d(Tx,Ty) \le k\{d(x,Tx)+d(y,Ty)\} \tag{1.2}$$

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for all $x, y \in C$.

(iii) Chatterjea type mapping [5] if there exists a number $k \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq k\{d(x, Ty) + d(y, Tx)\} \tag{1.3}$$

for all $x, y \in C$.

(iv) Reich type mapping [30] if there exist nonnegative numbers a, b, c satisfying $a + b + c < 1$ such that

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) \tag{1.4}$$

for all $x, y \in C$.

(v) Hardy and Rogers type mapping [14] if there exist nonnegative numbers a_i $(i = 1, 2, \cdots, 5)$ satisfying $\sum_{i=1}^{5} a_i < 1$ such that

$$\begin{aligned} d(Tx, Ty) \quad \leq \quad & a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) \\ & + a_4 d(x, Ty) + a_5 d(y, Tx) \end{aligned} \tag{1.5}$$

for all $x, y \in C$.

(vi) Zamfirescu type mapping [40] if there exist real numbers a, b, c satisfying $0 < a < 1, 0 < b, c < \frac{1}{2}$ such that for all pair of $x, y \in C$ at least one of the following is true:

(z_1) $d(Tx, Ty) \leq ad(x, y)$,

(z_2) $d(Tx, Ty) \leq b\{d(x, Tx) + d(y, Ty)\}$,

(z_3) $d(Tx, Ty) \leq c\{d(x, Ty) + d(y, Tx)\}$.

(vii) Quasi-contractive mapping [11] if there exists a constant $0 \leq k < 1$ such that

$$\begin{aligned} d(Tx, Ty) \quad \leq \quad & k \max\{d(x, y), d(x, Tx), d(y, Ty), \\ & d(x, Ty), d(y, Tx)\} \end{aligned} \tag{1.6}$$

for all $x, y \in C$.

(viii) Gregus type mapping [13] if there exists a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \quad \leq \quad kd(x, y) + (1 - k) \max\{d(x, Tx), d(y, Ty)\} \tag{1.7}$$

for all $x, y \in C$.

(x) Nearly Lipschitzian mapping [37] if for a fix sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$, there exists a constant $k_n \geq 0$ such that

$$d(T^n x, T^n y) \leq k_n\{d(x, y) + a_n\} \tag{1.8}$$

for all $x, y \in C$ and $n \in N$.

(xi) $T \in S(a, L)$ [38] if there exist constants $a \in (0, 1)$ and $L \geq 0$

satisfying

$$d(Tx, Ty) \leq ad(x, y) + L \max\{d(x, Tx), d(y, Ty)\} \tag{1.9}$$

for all $x, y \in C$.

Banach [1] introduced the first contractive definition in a complete metric space in the year 1922, which is known as Banach contraction principle. It deals with the dual question in fixed point theory:

(1) **Existence and Uniqueness**: Equation $Tx = x$ has exactly one solution i.e., T has one fixed point in C.

In 1968, Kannan [19] proved a fixed point theorem for discontinuous mapping. Following Kannan's paper, a lot of papers are devoted to obtain fixed points for various classes of contractive type mappings, that do not require the continuity of mappings for example,(see [3]–[4], [5], [11], [13], [19], [20], [30], [36], [37]–[38]). Although the mappings appearing in these papers are more general that either Banach's or Kannan's contractive mappings. Note that although condition (1.1) implies the continuity of the mapping T, condition (1.2) to (1.9) may hold even if the mapping is not continuous.

(2) **Convergence of Iteration**: Banach contraction principle also gives a constructive procedure for obtaining better and better approximations to the fixed point. This procedure is called an iteration process. By definition, this is a method such that we choose an arbitrary x_0 in a given set X and calculate recursively a sequence $x_1, x_2, \cdots$ from a relation of the form

$$x_{n+1} = Tx_n \quad n = 0, 1, 2, \cdots . \tag{1.10}$$

The iteration procedure (1.10) is commonly known as Picard iteration. The Picard iteration process can be used to approximate the unique fixed point for contraction mapping.

On the other hand, the following fixed point iteration processes have been extensively studied by many authors for approximating either the fixed points of nonlinear mappings (when these mappings are already known to have fixed points) or solution of nonlinear operator equations (see e.g. [3]–[10], [18], [21], [31], [32]):

(MS) Mann iteration process (see [21]) is defined as follows: For C a convex subset of a Banach space X and T a nonlinear mapping of C into itself, the sequence $\{x_n\}$ is generated from $x_0 \in C$ is defined by:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \ n \geq 0, \tag{1.11}$$

where $\{\alpha_n\}$ is real sequence in $[0, 1]$ which satisfies the conditions:

(A$_1$) $0 \leq \alpha_n < 1, \ n \geq 0,$

(A$_2$) $\sum_{n=0}^{\infty} \alpha_n = \infty.$

In some application's condition (A_2) is replaced by $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty.$

(IS) Ishikawa iteration process (see [18]) is defined as follows: With C and T as in (1.11) the sequence $\{x_n\}$ is generated from $x_0 \in C$ is defined by:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \ n \geq 0, \end{aligned} \tag{1.12}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$ which satisfies the conditions:

(B$_1$) $0 \leq \alpha_n \leq \beta_n < 1,$

(B$_2$) $\lim_{n\to\infty} \beta_n = 0,$

(B$_3$) $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty.$

The iteration schemes (1.11) and (1.12) may exhibits different behaviors for different classes of nonlinear mappings (see [32]).

Recently, Sahu [36] introduced the generalized Ishikawa iteration process of rank r as below:

(GIS) Let C be a nonempty subset of a normed space X and $T : C \to X$ be a nonlinear operator. Further, let r be a positive integer and let $\{a_{n,i}\}$, $i = 1, 2, 3, \cdots, r$ be a real sequence in $[0,1]$.

For $x_0 \in C$, the generalized Ishikawa iterative sequence (of rank r) $\{x_n\}_{n=0}^{\infty}$ is given by:

$$\begin{aligned} x_{n+1} &= (1-a_{n,1})x_n + a_{n,1} T y_{n,1} \\ y_{n,i} &= (1-a_{n,i+1})x_n + a_{n,i+1} T y_{n,i+1}, \quad i = 1, 2, \cdots, r-1, \\ y_{n,r} &= x_n \ \forall\ n \geq 0, \end{aligned} \tag{1.13}$$

In particular, we underline that whenever referring to three steps iteration process mean the procedure defined for rank 3 and defined as follows:

For $x_0 \in C$

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \quad \forall\ n \geq 0, \end{aligned} \tag{1.14}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $[0,1]$.

In particular, if we put $\gamma_n = 0$ in (1.14) then the sequence is the Ishikawa iteration process (1.12) and we put $\beta_n = \gamma_n = 0$ in (1.14) then the sequence is the Mann iteration process (1.11).

On the other hand, Harder and Hicks [17] mentioned that the study of stability of iterative scheme is useful in both theoretical and numerical investigation.

Definition 1.1. *Let X be an arbitrary Banach space and C be a nonempty subset of X. Let $T : C \to C$ be a self mapping and $F(T)$ is a set of fixed point which is nonempty i.e. $F(T) = \{x : Tx = x\} \neq \phi$. A mapping $f(T, \cdot) : C \to C$ is said to be an iterative process (or scheme), when $f(T, \cdot)$ is considered as a procedure, involving T, which yields a sequence of points $\{x_n\} \subset C$ defined by:*

$$x_{n+1} = f(T, \cdot), \qquad \forall n \geq 0, \tag{1.15}$$

where $x_0 \in C$ is given.

Example 1.2. *By setting $f(T, \cdot) = Tx_n$ then (1.15) reduces to the Picard iteration scheme. The iteration scheme also includes the Mann (1.11), the Ishikawa (1.12) and the generalized Ishikawa iteration process of rank 3, (1.14).*

Definition 1.3. *Let X be an arbitrary Banach space and C be a nonempty subset of X. Let $T : C \to C$ be a self mapping and $F(T)$ is a set of fixed point which is nonempty i.e. $F(T) = \{x : Tx = x\} \neq \phi$. Consider the fixed point iteration process defined by (1.15). Suppose that the sequence $\{x_n\}$ converges strongly to a fixed point $p \in F(T)$. Let $\{y_n\}$ be any sequence in C and define by*

$$\epsilon_n = \|y_{n+1} - f(T, y_n)\| \qquad \forall n \geq 0.$$

If $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} y_n = p$, then iterative scheme (1.15) is said to be T-stable (see [17]).

We say that an iterative scheme (1.15) is almost T-stable or almost stable with respect to T if $\sum_{n=0}^{\infty} \|y_{n+1} - f(T, y_n)\| < \infty$ implies that $y_n \to p$ as $n \to \infty$ (see [28]).

It is clear that iterative scheme $f(T, \cdot)$ which is T-stable is almost T-stable, but converse is not true (see [28]).

Stability of iterative scheme for various type of nonlinear mappings has been studied during last two decades (see e.g. [2], [16], [22]–[29], [33]–[35]).

In this view, we have the following natural question:

Question 1.4. *Is it possible to discuss approximation and stability result for generalized Ishikawa iteration process of rank* 3 *for* $S(a,b,c)$*-contraction (see the Definition 2.2) in a Banach space?*

In this paper, we give an affirmative answer of Question 1.4. We established existence, convergence and stability theorem for $S(a,b,c)$-contraction by using generalized Ishikawa iteration process of rank 3 in a Banach space. Our result extends the result of Banach [1], Kannan [19] and Lucimer [20] and many others in these directions.

2. Preliminaries

For our main result we need the following definitions and lemmas.

Definition 2.1. *Let* C *be a nonempty subset of a Banach space* X*, and* T *be a self mapping from* C *into itself such that*

$$\|Tx - Ty\| \leq a\|x-y\| + b[\|x-Tx\| + \|y-Ty\|] \tag{2.1}$$

for $x, y \in C$*, where* $0 \leq a, b \leq 1$*. Any mapping* T *satisfying (2.1) is said to be* $T \in D(a,b)$ *(see* [20]*).*

Definition 2.2. *Let* C *be a nonempty subset of a Banach space* X*, and* T *be a self mapping from* C *into itself such that* T *satisfies the inequality*

$$\|Tx - Ty\| \leq a\|x-y\| + b\|x-Tx\| + c\|y-Ty\| \tag{2.2}$$

for $x, y \in C$*, and* $a, b, c,$ *are nonnegative real numbers such that* $a+b+c<1$*. A mapping* T *satisfying (2.2) is said to be an* $S(a,b,c)$*-contraction or* $T \in S(a,b,c)$*.*

The following example support the definition 2.2, but it is not a contraction.

Example 2.3. *Let* $H = (-\infty, \infty)$ *with the usual norm and* $C = [0,1]$*. Define* $T : C \to C$ *by*

$$Tx = \begin{cases} \frac{1}{2}, & for \quad x \in [0,1) \\ \frac{1}{4} & for \quad x = 1. \end{cases}$$

It is easy to see that $T \in S(a,b,c)$ *with* $a = \frac{1}{2}, b = \frac{1}{4}, c = \frac{1}{5}$ *for all* $x, y \in C$*.*

Lemma 2.4. *(Weng* [39]*) Let* $\{\psi_n\}$ *be a nonnegative real sequence such that*

$$\psi_{n+1} \leq (1-\xi_n)\psi_n + \sigma_n$$

where $\xi_n \in [0,1]$ *and* $\sum_{n=0}^{\infty} \xi_n = \infty$ *and* $O(\xi_n) = \sigma_n$*. Then* $\psi_n \to 0$ *as* $n \to \infty$*.*

Lemma 2.5. *Let C be a nonempty subset of a Banach space X, and $T \in S(a,b,c)$ such that $a+b+c<1$. If $F(T)=\{x : x=Tx\} \neq \phi$, then $F(T)$ is a singleton.*

Proof. Let $x_1, x_2 \in F(T)$ such that $x_1 \neq x_2$. Since T satisfy (2.2), we have

$$\begin{aligned}\|x_1-x_2\| &= \|Tx_1-Tx_2\| \\ &\leq a\|x_1-x_2\| + b\|x_1-Tx_1\| + c\|x_2-Tx_2\|.\end{aligned}$$

Therefore, we have

$$\|x_1-x_2\| \leq a\|x_1-x_2\|$$

which is a contradiction. Therefore, $F(T)$ is a singleton. □

Lemma 2.6. *Let C be a nonempty subset of a Banach space X and $T \in S(a,b,c)$ with $a+2b<1$. Then*

$$\|Tx-p\| \leq \delta\|x-p\|$$

where $\delta = \frac{a+b}{1-b}$, $\forall x \in C$ and $p \in F(T)$.

Proof. From (2.2), we have

$$\begin{aligned}\|Tx-Tp\| &\leq a\|x-p\| + b\|x-Tx\| + c\|p-Tp\| \\ &\leq a\|x-p\| + b[\|x-p\| + \|p-Tx\|],\end{aligned}$$

which implies that

$$\begin{aligned}\|Tx-Tp\| &\leq \frac{a+b}{1-b}\|x-p\| \\ &= \delta\|x-p\|,\end{aligned}$$

where $\delta = \frac{a+b}{1-b} < 1$. Hence, we have

$$\|Tx-p\| \leq \delta\|x-p\|,$$

for all $x \in C$ and $p \in F(T)$. □

Theorem 2.7. *Let C be a nonempty closed subset of a Banach space X and $T \in S(a,b,c)$ with $a+b+c<1$. Then T has a unique fixed point.*

Proof. For $x_0 \in C$, let $\{x_n\}$ be a sequence defined by (1.10) and $T \in S(a,b,c)$, then by (2.2),we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|Tx_n - Tx_{n-1}\| \\ &\leq a\|x_n-x_{n-1}\| + b\|x_{n-1}-Tx_{n-1}\| + c\|x_n-Tx_n\| \\ &\leq a\|x_n-x_{n-1}\| + b\|x_{n-1}-x_n\| + c\|x_n-x_{n+1}\|,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \frac{(a+b)}{(1-c)}\|x_n-x_{n-1}\| \\ &= k\|x_n-x_{n-1}\|, \end{aligned} \tag{2.3}$$

where $k=\frac{a+b}{1-c}<1$ and by induction, we have

$$\|x_{n+1}-x_n\| \le k^n\|x_1-x_0\|, \qquad n=0,1,2,\cdot.$$

Thus for any numbers $n,m \in N, m>0$, we have

$$\begin{aligned}\|x_{n+m}-x_n\| &\le \sum_{j=n}^{m+n-1}\|x_{j+1}-x_j\| \\ &\le \sum_{j=n}^{m+n-1}k^j\|x_1-x_0\| \\ &\le \frac{k^n}{1-k}\|x_0-x_1\|. \end{aligned} \tag{2.4}$$

Since $0 \le k < 1$, it results that $\frac{k^n}{1-k} \to 0$ as $n \to \infty$, which together with (2.4) shows that sequence $\{x_n\}$ is a Cauchy sequence in C. Since C is closed subset of a Banach space X, therefore $\{x_n\}$ converges to some $p \in C$. Also, we know that

$$\begin{aligned}\|p-Tp\| &\le \|x_{n+1}-p\|+\|x_{n+1}-Tp\| \\ &\le \|x_{n+1}-p\|+\|Tx_n-Tp\| \\ &\le \|x_{n+1}-p\|+a\|x_n-p\|+b\|x_n-Tx_n\|+c\|p-Tp\|, \end{aligned}$$

which implies that

$$\|p-Tp\| \le \frac{1+b}{1-c}\|x_{n+1}-p\|+\frac{a+b}{1-c}\|x_n-p\|.$$

Letting $n \to \infty$, then we have $Tp=p$. Hence T has a fixed point. Uniqueness of fixed point follows from Lemma 2.5. □

3. Main Result

In this section, we prove convergence and stability of the generalized Ishikawa iterative process of rank 3 in Banach space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Banach space X and $T \in S(a,b,c)$ with $a+2b<1$. Suppose that $F(T)=\{x : x=Tx\} \neq \phi$ and $x_0 \in C$ be arbitrary. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $[0,1]$ satisfying the following condition:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Let $\{x_n\}$ be a sequence in C defined by (1.14) and $\{f_n\}$ be an arbitrary sequence in C defined by

$$\epsilon_n = \|f_{n+1} - (1-\alpha_n)f_n + \alpha_n T s_n\| \tag{3.1}$$

$$\begin{aligned} s_n &= (1-\beta_n)f_n + \beta_n T u_n, \\ u_n &= (1-\gamma_n)f_n + \gamma_n T f_n, \quad \forall\, n \geq 0. \end{aligned} \tag{3.2}$$

Then we have the following:

(a) *$\{x_n\}$ converges strongly to a unique fixed point p of T.*

(b) *$\|f_{n+1} - p\| \leq [1 - \alpha_n(1-\delta)]\|f_n - p\| + \epsilon_n$.*

(c) *$\lim_{n\to\infty} f_n = p$ if and only if $\lim_{n\to\infty} \epsilon_n = 0$.*

Proof. (a) By Theorem 2.7, T has a unique fixed point say $p \in C$. Using Lemma 2.6 and (1.14), we have

$$\begin{aligned} \|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n T x_n - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\delta\|x_n - p\| \\ &\leq [1 - \gamma_n(1-\delta)]\|x_n - p\|. \end{aligned} \tag{3.3}$$

Using Lemma 2.6, (1.14) and (3.3), we have

$$\begin{aligned} \|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n T z_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|Tz_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\delta\|z_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| \\ &+ \beta_n\delta[1 - \gamma_n(1-\delta)]\|x_n - p\| \\ &\leq [1 - \beta_n + \beta_n\delta(1 - \gamma_n(1-\delta))]\|x_n - p\|. \end{aligned} \tag{3.4}$$

Again from Lemma 2.6, (1.14) and (3.4), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)x_n + \alpha_n T y_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|Ty_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\delta\|y_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| \\ &+ \alpha_n\delta[1 - \beta_n + \beta_n\delta(1 - \gamma_n(1-\delta))]\|x_n - p\| \\ &\leq \Big[1 - \alpha_n(1-\delta)[1 + \beta_n\delta + \beta_n\gamma_n\delta^2]\Big]\|x_n - p\|. \end{aligned} \tag{3.5}$$

It may be noted that for $\delta \in [0,1)$ and $\eta_n \in [0,1]$, the following inequality is always true

$$1 \leq 1 + \delta\eta_n \leq 1 + \delta. \tag{3.6}$$

Using (3.5) and (3.6), we have

$$\|x_{n+1} - p\| \leq (1 - \alpha_n(1-\delta))\|x_n - p\|. \tag{3.7}$$

Set $\psi_n = \|x_n - p\|$ and $\xi_n = (1 - \alpha_n(1-\delta))$ for all $n \geq 0$. Using Lemma 2.4, (i) and (3.7), we conclude immediately that $\psi_n \to 0$ as $n \to \infty$. Therefore, sequence $\{x_n\}$ converges strongly to unique fixed point of T. This completes the proof of (a).

(b) Using (3.1), we have

$$\begin{aligned} \|f_{n+1} - p\| &= \|f_{n+1} - (1-\alpha_n)f_n + \alpha_n T s_n + (1-\alpha_n)f_n + \alpha_n T s_n - p\| \\ &\leq \|(1-\alpha_n)f_n + \alpha_n T s_n - p\| + \epsilon_n. \end{aligned} \tag{3.8}$$

Set $P_n = (1-\alpha_n)f_n + \alpha_n T s_n$. Using Lemma 2.6 and (3.2), we have

$$\begin{aligned} \|u_n - p\| &= \|(1-\gamma_n)f_n + \gamma_n T f_n - p\| \\ &\leq (1-\gamma_n)\|f_n - p\| + \gamma_n\|T f_n - p\| \\ &\leq (1-\gamma_n)\|f_n - p\| + \gamma_n\delta\|f_n - p\| \\ &\leq [1 - \gamma_n(1-\delta)]\|f_n - p\|. \end{aligned} \tag{3.9}$$

Using Lemma 2.6, (3.2) and (3.9), we have

$$\begin{aligned} \|s_n - p\| &= \|(1-\beta_n)f_n + \beta_n T u_n - p\| \\ &\leq (1-\beta_n)\|f_n - p\| + \beta_n\|T u_n - p\| \\ &\leq (1-\beta_n)\|f_n - p\| + \beta_n\delta\|u_n - p\| \\ &\leq (1-\beta_n)\|f_n - p\| \\ &+ \beta_n\delta[1 - \gamma_n(1-\delta)]\|f_n - p\| \\ &\leq [1 - \beta_n + \beta_n\delta(1 - \gamma_n(1-\delta))]\|f_n - p\|. \end{aligned} \tag{3.10}$$

Again from Lemma 2.6, (3.2) and (3.10), we have

$$\begin{aligned} \|P_n - p\| &= \|(1-\alpha_n)f_n + \alpha_n T s_n - p\| \\ &\leq (1-\alpha_n)\|f_n - p\| + \alpha_n\|T s_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\delta\|s_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| \\ &+ \alpha_n\delta[1 - \beta_n + \beta_n\delta(1 - \gamma_n(1-\delta))]\|f_n - p\| \\ &\leq \Big[1 - \alpha_n(1-\delta)[1 + \beta_n\delta + \beta_n\gamma_n\delta^2]\Big]\|f_n - p\|. \end{aligned} \tag{3.11}$$

From (3.6) and (3.11), we have

$$\|P_n - p\| \leq (1-\alpha_n(1-\delta))\|f_n - p\|. \tag{3.12}$$

From (3.8) and (3.12), we have

$$\|f_{n+1} - p\| \leq (1-\alpha_n(1-\delta))\|f_n - p\| + \epsilon_n. \tag{3.13}$$

This completes the proof of (b).

(c) Suppose $\lim\limits_{n\to\infty} f_n = p$. Using triangle inequality, (3.1) and (3.12), we infer that

$$\begin{aligned}\epsilon_n &= \|f_{n+1} - P_n\| \\ &\leq \|f_{n+1} - p\| + \|P_n - p\| \\ &\leq \|f_{n+1} - p\| + (1-\alpha_n(1-\delta))\|f_n - p\| \to 0 \quad as \quad n \to \infty.\end{aligned}$$

Conversely, suppose that $\lim\limits_{n\to\infty} \epsilon_n = 0$. Set $\psi_n = \|f_n - p\|$ and $\xi_n = (1-\alpha_n(1-\delta))$ for all $n \geq 0$ and $\sigma_n = \epsilon_n$. Using Lemma 2.4, (i) and (3.13), we conclude immediately that $\psi_n \to 0$ as $n \to \infty$. Therefore, sequence $\{f_n\}$ converges strongly to unique fixed point of T. This completes the proof of (c). □

Remark 3.2. *Theorem 3.1 provides an affirmative answer of question 1.4.*

Remark 3.3. (1) *A contraction mapping (1.1) is in a class* $S(a,0,0)$, *hence Theorem 3.1 is true for contraction mapping.*
(2) *The Kannan type mapping (1.2) is in a class* $S(0,b,c)$, *where* $b = c \in (0, \frac{1}{2})$, *hence Theorem 3.1 is true for Kannan type mapping.*
(3) $T \in D(a,b)$ *is in a class* $S(a,b,b)$, *hence Theorem 3.1 is true for* $T \in D(a,b)$.

References

[1] S. Banach, *Sur les operations dans les ensembles abstrains et leur application aux equation integral*, Fund. Math. **3** (1922), 133–181.

[2] V. Berinde, *On the stability of some fixed point procedures*, Bul. Stiint Univ. Baia Mare, **XVII**, (2002), 7–14,

[3] V. Berinde, *On the convergence of the Ishikawa iteration in the class of quasi-contrcative operators*, Acta. Math. Univ. Comeniance, Vol. **LXXII**, (2004), 1–11.

[4] V. Berinde, *Iterative approximation of fixed points*, Editura Efemeride, Baia Mare, (2002).

[5] S. K. Chatterjee, *Fixed point theorems*, C.R. Acad. Bulgare Sci. **25** (1972), 727–730.

[6] C. E. Chidume, *On the Ishikawa fixed point iteration for quasi-contractive mappings*, J. Nigerian Math. Soc. **4** (1985), 1–11.

[7] C. E. Chidume, *On the Ishikawa and Mann iteration methdos for quasi-contrcative mappings*, J. Nigerian Math. Soc. **7** (1988), 1–9.

[8] C. E. Chidume, *Fixed point iterations for certian classes of nonlinear mappings*, Applicable Anal. **27 1-3** (1988), 31–45.

[9] C. E. Chidume, *Approximation of fixed points of quasi-contractive mappings in L_p Spaces*, Indian J. Pure. Appl. Math. **22(4)** (1991), 273–286.

[10] C. E. Chidume and M. O. Osilike, *Fixed point iterations for quasi-contractive maps in uniformly smooth Banach spaces*, Bull. Korean Math. Soc. **30 (2)**(1993), 201–212.
[11] L. B. Ciric, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1975), 267–273.
[12] Jr. W. G. Doston, *On the Mann iterative process*, Trans. Amer. Math. Soc. **149** (1970), 65–73.
[13] M. Gergus, *A fixed point theorem in Banach space*, Boll. Un. Mat. Ital. **5** (1980), 193–198.
[14] G. E. Hardy and T. D. Rogers, *A generalization of fixed point theorem of Reich*, Canad. Math. Bull. **16** (1973), 201–206.
[15] A. M. Harder, *Fixed point theory and stability results for fixed point iteration procedure*, Ph.D thesis, Univ. of Missouri-Rolla, 1987.
[16] A. M. Harder and T. L. Hicks, *A stable iteration procedure for nonexpansive mapping*, Math. Japonica, **33(5)** (1988), 687–692.
[17] A. M. Harder and T. L. Hicks, *Stability results for fixed point iterations procedure*, Math. Japonica, **33(5)** (1988), 693–706.
[18] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44(1)** (1974), 147–150.
[19] R. Kannan, *Some results on fixed points*, Bul. Cal. Math. Soc. **10** (1968), 71–76.
[20] N. G. Lucimar, *Fixed point theoreoms for some discontinuous operators*, Pacific J. Math. **123(1)** (1986), 189–196.
[21] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.
[22] M. O. Osilike, *Stability results for fixed point iteration procedures*, J. Nigerian Math. Soc. **14** (1995), 441–449.
[23] M. O. Osilike, *Stability results for the Ishikawa fixed point iteration procedure*, Indian J. Pure and Applied Math. **26** (1995), 937–945.
[24] M. O. Osilike, *Stable iteration procedure for quasi-contractive maps*, Indian J. Pure and Appl. Math. **27** (1996), 25–34.
[25] M. O. Osilike, *Stable iteration procedures for strong pseudo-contrations and nonlinear equations of accretive type*, J. Math. Anal. Appl., **204** (1996), 676–692.
[26] M. O. Osilike, *Stability of the Ishikawa ietartion method for quasi-contractive maps*, Indian J. Pure and Appl. Math. **28(9)** (1997), 1251–1265.
[27] M. O. Osilike, *Stable iteration procedure for nonlinear pseudo-contractive and accretive operators in arbitrary Banach spaces*, Indian J. Pure and Appl. Math. **28(8)** (1997), 1017–1029.
[28] M. O. Osilike, *Stability of the Mann and Isshikawa iteration procedure for ϕ-strongly pseudo-contractions and nonlinear equations of the ϕ- strongly accretive type*, J. Math. Anal. Appl. **227** (1998), 319–334.
[29] M. O. Osilike and A. Udomene, *Short proofs of stability result for fixed point iteration procedure for a class of contractive type mappings*, Indian J. Pure and Appl. Math. **30(2)** (1999), 1229–1234.
[30] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull. **14** (1971), 121–124.
[31] B. E. Rhoades, *Fixed point iterations using infinite matrices*, Trans. Amer. Math. Soc. **196** (1974), 161–176.
[32] B. E. Rhoades, *Comments on two fixed point iteration methods*, J. Math. Anal. Appl. **56 (2)** (1976), 741–750.

[33] B. E. Rhoades, *Fixed point theorems and stability results for fixed point iteration procedure*, Indian J. Pure Appl. Math. **21** (1990), 1–9.

[34] B. E. Rhoades, *Some fixed point iteration procedure*, Int. J. Math and Math. Sci. **14** (1991), 1–16.

[35] B. E. Rhoades, *Fixed point theorems and stability result for fixed point iteration procedure -II*, Indian J. Pure and Appl. Math. **24** (1993), 691–703.

[36] D. R. Sahu, *On generalized Ishikawa iteration process and nonexpansive mapping in Banach spaces*, Demonstraio Mathematica **36(3)** (2003), 721–734.

[37] D. R. Sahu, *Fixed points demicontinuous neraly Lipschitzian mapping in Banach spaces*, Comment Math. Univ. Carolinacs **46(4)** (2005), 653–666.

[38] D. R. Sahu and Samir Dashputre, *Approximation of fixed points of new class of mappings*, Nonlinear Functional Analysis and Application **12(3)** (2007), 387–397.

[39] X. L. Weng, *Fixed point iteration for local strictly pseudocontractive mappings*, Proc. Amer. Math. Soc. **113** (1991), 727–731.

[40] T. Zamfriescu, *Fixed point theorems in metric spaces*, Arch. Math. (Basel) **23** (1972), 292–298.