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INSTABILITY OF NON-NEGATIVE SOLUTIONS FOR A BOUNDARY VALUE PROBLEM WITH A INDEFINITE WEIGHT FUNCTION

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Abstract. We consider the stability of non-negative solution of the following equation:

$$\begin{cases} -\triangle(au) = & \lambda m(x)u(x)(u(x)-1) \qquad x \in \Omega, \\ \quad Bu = & 0, \qquad\qquad\qquad\qquad\quad x \in \partial\Omega. \end{cases}$$

If $\triangle(au) = div\nabla(au)$, Ω is a bounded domain in R^N $(N \geq 1)$ with smooth boundary $Bu(x) = \alpha h(x)u + (1-\alpha)\frac{\partial u}{\partial n}$ for $\alpha \in [0.1]$, $h : \partial\Omega \to \mathbb{R}^+$ with $h = 1$ when $\alpha = 1$, $\lambda > 0$, then the weight $m(x)$ satisfies $m(x) \in C(\Omega)$, $m(x) \geq m_0 > 0$ for all $x \in \Omega$ and $a : \Omega \to \mathbb{R}^+$ satisfy certain conditions which guaranty the existence of the weak solution of this equation. We prove that every nontrivial non-negative solution is unstable under certain conditions.

1. Introduction

In this paper, we study the stability of nontrivial non-negative solutions to the elliptic boundary value problem

$$-\triangle(au) = \lambda m(x)u(x)(u(x)-1) \qquad x \in \Omega, \tag{1.1}$$

$$Bu = 0, \qquad x \in \partial\Omega, \tag{1.2}$$

where $\triangle(au) = div\nabla(au)$, Ω is a bounded domain in R^N $(N \geq 1)$ with smooth boundary

$$Bu(x) = \alpha h(x)u + (1-\alpha)\frac{\partial u}{\partial n}$$

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for $\alpha \in [0,1]$, $h : \partial\Omega \to \mathbb{R}^+$ with $h = 1$ when $\alpha = 1$, i.e., the boundary condition may be of Dirichlet, Neumann or mixed type. $\lambda > 0$ is a constant, the weight $m(x)$ satisfies $m(x) \in C(\Omega)$, $m(x) \geq m_0 > 0$ for all $x \in \Omega$, and a satisfy certain conditions which guaranty the existence of the weak solution of (1.1)-(1.2).

This problem (without weight $m(x)$, $a(x)$) was studied by several authors. Shivaji and co-authors have shown that every nontrivial solution of (1) with Dirichlet boundary condition $u \mid \partial\Omega = 0$ is unstable[4]. Moreover in this case Tertikas [7] proved it by using sub and super-solution. In [4] the corresponding equation with p-Laplacian is studied. Also Afrouzi and Rasouli in [2] have studied this problem (with indefinite weight and $a(x) = 1$).

The purpose of this paper is to extend under certain conditions but for $\triangle(au)$. We shall prove the instability of nontrivial non-negative solution u by showing that the principle eigenvalue μ_1(see [1]), of the corresponding linearized equation about u is negative. Then the instability of u follows from the well-known principle of linearized stability (see [6]). We shall prove the instability of positive solution u by showing that the principle eigenvalue μ_1 of the linearized equation about u is negative. Instability of u then follows from the well-known principle of linearized stability (see [5]).

Definition 1.1. *a) The equation* (1) *holds in the weak sense for some* $u \in X$ *and* $\lambda > 0$ *and* $m(x) \geq m_0 > 0$ *if the following integral identity*

$$\int_\Omega \nabla(au)\nabla v dx = \lambda \int_\Omega [m(x)u(x)(u(x)-1)]v dx$$

is fulfilled for any $v \in X$.

b) A solution u *of* (1) *is called linearly stable if the principle eigenvalue* μ_1 *of its linearization is positive, otherwise is linearly unstable. For a solution we require that* $u \in C^1(\overline{\Omega})$ *satisfies in* $\nabla u \in C^1(\Omega)$.

2. Main result

We now state our main result, i.e., instability of nonnegative solutions.

Lemma 2.1. *Suppose that* $[\triangle(au)]_v$ *is Gateaux differential of* $\triangle(au)$ *in direct of* v. *Then*

$$[\triangle(au)]_v = \triangle(av) \tag{2.1}$$

Proof. First, we note that

$$\begin{aligned} F(u) &= \triangle(au) \\ &= \nabla.\nabla(au) \\ &= \nabla(a\nabla u + u\nabla a) \\ &= \nabla(a\nabla u) + \nabla(u\nabla a) \\ &= \nabla a\nabla u + a\triangle u + \nabla u\nabla a + u\triangle a \\ &= 2\nabla a\nabla u + a\triangle u + u\triangle a. \end{aligned}$$

Then,

$$F(u+tv) = 2\nabla a(\nabla u + t\nabla v) + a(\triangle u + t\triangle v) + (u+tv)\triangle a$$

so,

$$\begin{aligned} \frac{d}{dt}F(u+tv)\mid_{t=0} &= 2\nabla a\nabla v + a\triangle v + v\triangle a \\ &= \triangle(av) \end{aligned}$$

□

Theorem 2.2. *If $\nabla a\nabla v > 0$ for any $v \in X$, then every nontrivial solution of (1.1)-(1.2) is linearly unstable.*

Proof. Let u_0 be any non-trivial non-negative solution of (1.1)-(1.2), $f(u(x)) = u(x)(u(x)-1)$, $g(u(x)) = f(u(x)) - f(0) + \mid f'(0) \mid u(x) = u^2(x)$. Then $g(0) = 0$, $g'(u) = 2u$, $g''(u) = 2 > 0$. Therefore, $g'(u) > 0$ for $u > 0$ and $g(u) > 0$ for $u > 0$. Now (1.1)-(1.2) can be rewritten as

$$-\triangle(au) = \lambda m(x)\{g(u(x)) - u(x)\}, \qquad x \in \Omega, \tag{2.2}$$

$$Bu(x) = 0, \qquad x \in \partial\Omega. \tag{2.3}$$

Let u_0 be any nontrivial non-negative solution of (2.2), (2.3). Then the linearized equation about u_0 is

$$-\triangle(a\phi) - \lambda m(x)\{g'(u_0) - 1\}\phi = \mu\phi \qquad x \in \Omega, \tag{2.4}$$

$$Bu(x) = 0, \qquad x \in \partial\Omega. \tag{2.5}$$

Let μ_1 be the principle eigenvalue and $\psi(x)(\geq 0)$ be a corresponding eigenfunction. Multiplying (2.2) $g'(u_0)\psi(x) -$ (2.4)$g(u_0)$ and integrating over Ω, we obtain

$$\begin{aligned} -\mu_1 \int_\Omega \psi(x)u_o^2 dx &= \int_\Omega [u_o^2\triangle(a\psi) - 2u_o\psi(x)\triangle(au_o)]dx \\ &\quad + \lambda\int_\Omega m(x)\psi(x)u_o^2 dx \end{aligned} \tag{2.6}$$

But by *Green's* first identity

$$\begin{aligned}\int_\Omega u_o^2\triangle(a\psi)dx &= \int_{\partial\Omega} u_o^2\frac{\partial(a\psi)}{\partial n}ds - \int_\Omega 2u_o\nabla u_o\nabla(a\psi)dx \qquad (2.7)\\ &= \int_{\partial\Omega} u_o^2\frac{\partial(a\psi)}{\partial n}ds - \int_\Omega 2u_o a\nabla u_o\nabla\psi dx\\ &\quad -\int_\Omega 2u_o\psi\nabla u_o\nabla a dx,\end{aligned}$$

and

$$\begin{aligned}\int_\Omega 2u_o\psi\triangle(au_o)dx &= \int_{\partial\Omega} 2u_o\psi\frac{\partial(au_o)}{\partial n}ds - \int_\Omega 2\nabla(u_o\psi)\nabla(au_o)dx \qquad (2.8)\\ &= \int_{\partial\Omega} 2u_o\psi\frac{\partial(au_o)}{\partial n}ds - \int_\Omega 2u_o\psi\nabla u_o\nabla a dx \qquad (2.9)\\ &\quad -\int_\Omega 2a\psi \mid \nabla u_o \mid^2 dx - \int_\Omega u_o^2\nabla a\nabla\psi dx\\ &\quad -\int_\Omega 2au_o\nabla u_o\nabla\psi dx.\end{aligned}$$

By using (2.7)-(2.8) in (2.6), we get

$$\begin{aligned}-\mu_1\int_\Omega \psi(x)u_o^2dx &= \lambda\int_\Omega m(x)\psi(x)u_o^2dx \qquad (2.10)\\ &\quad +\int_{\partial\Omega}[u_o^2\frac{\partial(a\psi)}{\partial n} - 2u_o\psi\frac{\partial(au_o)}{\partial n}]ds\\ &\quad +\int_\Omega 2a\psi \mid \nabla u_o \mid^2 dx\\ &\quad +\int_\Omega u_o^2\nabla a\nabla\psi dx.\end{aligned}$$

We notice that for $\alpha = 1$ (then $h = 1$), $Bu_0 = u_0 = 0$ on $s \in \partial\Omega$, so $g(u_o) = u_o = 0$ and $\psi = 0$ on $s \in \partial\Omega$. Hence,

$$\int_{\partial\Omega}[u_o^2\frac{\partial(a\psi)}{\partial n} - 2u_o\psi\frac{\partial(au_o)}{\partial n}]ds = 0. \qquad (2.11)$$

But for $\alpha \neq 1$,

$$\begin{aligned}\int_{\partial\Omega}[u_o^2\frac{\partial(a\psi)}{\partial n} - 2u_o\psi\frac{\partial(au_o)}{\partial n}]ds &= \int_{\partial\Omega}[u_o^2\frac{-\alpha ha\psi}{1-\alpha} - 2u_o\psi\frac{-\alpha ahu_0}{1-\alpha})]ds\\ &= \int_{\partial\Omega}(\frac{\alpha ha\psi(s)}{1-\alpha})u_o^2ds. \qquad (2.12)\end{aligned}$$

But $\alpha \geq 0$, $h > 0$,$\psi \geq 0$ for $s \in \partial\Omega$ and $u_o^2 > 0$. If $\alpha \neq 1$

$$\int_{\partial\Omega} [u_o^2 \frac{\partial(a\psi)}{\partial n} - 2u_o\psi \frac{\partial(au_o)}{\partial n}] ds \geq 0$$

also, since $\psi > 0$ on Ω,

$$\int_{\Omega} 2a\psi \mid \nabla u_o \mid^2 dx > 0.$$

Our assumption implies that

$$\int_{\Omega} u_o^2 \nabla\psi \nabla a dx \geq 0.$$

By (2.9),

$$-\mu_1 \int_{\Omega} \psi(x) u_o^2 dx > \lambda \int_{\Omega} m(x)\psi(x) u_o^2 dx.$$

Since $m(x) \geq m_0 > 0$ for $x \in \Omega$, we have

$$-\mu_1 \int_{\Omega} \psi(x) u_o^2 dx > \lambda \int_{\Omega} m_0 \psi(x) u_o^2 dx.$$

Hence,

$$-\mu_1 \int_{\Omega} \psi(x) u_0^2 dx > 0, \tag{2.13}$$

which implies that $\mu_1 < 0$ and the result follows from [5]. □

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