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COMMON FIXED POINT OF MULTI-STEP ITERATION SCHEME WITH ERRORS FOR FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we study sufficient and necessary condition for finite step iterative sequences with errors for finite family of asymptotically quasi-nonexpansive mappings in Banach space to converge to common fixed point. These results improve and extend the corresponding results of Ghosh and Debnath [4], Khan and Takahashi [6], Petryshyn and Williamson [8], Liu [9, 10], Shahzad and Udomene [14], Xu and Noor [17] and many others.

1. Introduction and Preliminaries

Let K be a nonempty subset of a real normed space E. Let T be a self mapping of K. Then T is said to be asymptotically nonexpansive with sequence $\{u_n\} \subset [0, \infty)$ if $\lim_{n\to\infty} u_n = 0$ and

$$\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\|$$

for all $x, y \in K$ and $n \geq 1$; and is said to be asymptotically quasi-nonexpansive with sequence $\{u_n\} \subset [0, \infty)$ if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$, $\lim_{n\to\infty} u_n = 0$ and

$$\|T^n x - x^*\| \leq (1 + u_n)\|x - x^*\|$$

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for all $x \in K, x^* \in F(T)$ and $n \geq 1$. The mapping T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$, and is called quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - x^*\| \leq \|x - x^*\|$$

for all $x \in K$ and $x^* \in F(T)$.

It is therefore clear that a nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive and an asymptotically nonexpansive mapping with a nonempty fixed point set is asymptotically quasi-nonexpansive but converse does not hold in general.

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [2] as an important generalization of the class of nonexpansive maps. They established that if K is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is an asymptotically nonexpansive self mapping of K, then T has a fixed point. In the year 1973, Goebel and Kirk [3] extended their own result to the broader class of uniformly L-Lipschitzian mappings with $L < \lambda$, where λ is sufficiently near 1.

Iterative techniques for approximating fixed points of nonexpansive mappings and their generalizations (asymptotically nonexpansive mappings etc.) have been studied by many authors (see e.g. Chidume [1], Rhoades [12], Schu [13], Tan and Xu [15]), using the Mann iteration process [7] or the Ishikawa iteration process [5].

In the year 1973, Petryshyn and Williamson [8] established a necessary and sufficient condition for the Mann iterative sequence to converge to a fixed point for quasi-nonexpansive mappings. Subsequently, Ghosh and Debnath [4] extended Petryshyn and Williamson's results and established some necessary and sufficient conditions for an Ishikawa-type iterative sequence to converge to a fixed point for quasi-nonexpansive mappings. Further, Qihou [9, 10, 11] extended the above results and established some sufficient and necessary conditions for Ishikawa iterative sequences or Ishikawa iterative sequences with errors for asymptotically quasi-nonexpansive mappings to converge to a fixed point. Recently, Shahzad and Udomene [14] established a sufficient and necessary conditions for the convergence of the Ishikawa type iterative sequences involving two asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings defined on a nonempty closed convex subset of a Banach space and a sufficient condition for the convergence of the Ishikawa type iterative sequences involving two uniformly continuous asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings defined on a nonempty closed convex subset of a uniformly convex Banach space.

Inspired and motivated by the above facts, we introduce and study a multi-step iterative scheme with errors for a finite family of asymptotically quasi-nonexpansive mappings. This scheme can be viewed as an extension for two step iterative schemes of Shahzad and Udomene [14]. The scheme may be defined as follows:

Let K be a nonempty closed convex subset of a Banach space E and let $T_1, T_2, \cdots, T_N\colon\ K \to K$ be asymptotically quasi-nonexpansive mappings, the following iteration scheme is studied:

$$\begin{aligned} x_n^1 &= \alpha_n^1 T_1^n x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1 \\ x_n^2 &= \alpha_n^2 T_2^n x_n^1 + \beta_n^2 x_n + \gamma_n^2 u_n^2 \\ &\cdots \\ x_{n+1} = x_n^N &= \alpha_n^N T_N^n x_n^{N-1} + \beta_n^N x_n + \gamma_n^N u_n^N \end{aligned} \tag{1}$$

with $x_1 \in K$, $n \geq 1$, where$\{\alpha_n^i\}$, $\{\beta_n^i\}$, $\{\gamma_n^i\}$, for all $1 \leq i \leq N$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, \cdots, N$, and $\{u_n^1\}, \{u_n^2\}, \cdots, \{u_n^N\}$ are bounded sequences in K.

For $N = 2$, $T_1 = T_2 = T$, $\beta_n = \alpha_n^1$, $\alpha_n = \alpha_n^2$ and $\gamma_n^1 = \gamma_n^2 = 0$, then (1) reduces to the scheme for a mapping defined by Liu [9]:

$$x_1 = x \in K$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n[(1-\beta_n)x_n + \beta_n T^n x_n], \qquad n \geq 1$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$.

For $N = 2$, $T_1, T_2\colon K \to K$, $T_1 = T$, $T_2 = S$, $\beta_n = \alpha_n^1$, $\alpha_n = \alpha_n^2$ and $\gamma_n^1 = \gamma_n^2 = 0$, then (1) reduces to the scheme for two mappings defined by Shahzad and Udomene [14]:

$$x_1 = x \in K$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n[(1-\beta_n)x_n + \beta_n T^n x_n], \qquad n \geq 1$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$.

For $N = 2$, $T_1, T_2\colon K \to K$, $T_1 = T_2 = T$, and $y_n = x_n^1$, then (1) reduces to the scheme with errors for a mapping defined by Liu [10]:

$$x_1 = x \in K$$

$$y_n = \alpha_n^1 T^n x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1$$
$$x_{n+1} = x_n^2 = \alpha_n^2 T^n y_n + \beta_n^2 x_n + \gamma_n^2 u_n^2$$

where $\{\alpha_n^1\}$, $\{\alpha_n^2\}$, $\{\beta_n^1\}$, $\{\beta_n^2\}$, $\{\gamma_n^1\}$, $\{\gamma_n^2\}$ are sequences in $[0,1]$ with $\alpha_n^1 + \beta_n^1 + \gamma_n^1 = 1 = \alpha_n^2 + \beta_n^2 + \gamma_n^2$ and $\{u_n^1\}$, $\{u_n^2\}$ are bounded sequences in K.

For $N = 2$, $T_1, T_2 \colon K \to K$, $T_1 = T$, $T_2 = S$ and $y_n = x_n^1$, then (1) reduces to the scheme with errors for two mappings defined by Shahzad and Udomene [14]:

$$\begin{aligned} x_1 &= x \in K \\ y_n &= \alpha_n^1 T^n x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1 \\ x_{n+1} = x_n^2 &= \alpha_n^2 S^n y_n + \beta_n^2 x_n + \gamma_n^2 u_n^2 \end{aligned}$$

where $\{\alpha_n^1\}$, $\{\alpha_n^2\}$, $\{\beta_n^1\}$, $\{\beta_n^2\}$, $\{\gamma_n^1\}$, $\{\gamma_n^2\}$ are sequences in $[0,1]$ with $\alpha_n^1 + \beta_n^1 + \gamma_n^1 = 1 = \alpha_n^2 + \beta_n^2 + \gamma_n^2$ and $\{u_n^1\}$, $\{u_n^2\}$ are bounded sequences in K.

The purpose of this paper is to establish a sufficient and necessary condition for strong convergence of the multistep iteration scheme with errors for a finite family of asymptotically quasi-nonexpansive mappings in Banach space. These results extend the related result of Shahzad and Udomene [14] and many others.

In order to prove our main results, the following lemmas have been used:

Lemma 1.1. *Let $\{a_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + r_n) a_n + \beta_n, \quad \forall n \in N.$$

Let $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. Then
(i) $\lim_{n\to\infty} a_n$ exists.
(ii) If $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2. ([16]) *Let $p > 1$ and $R > 1$ be two fixed numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g \colon [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda) y\|^p \leq \lambda \|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda) g(\|x - y\|)$$

for all $x, y \in B_R(0) = \{x \in E : \|x\| \leq R\}$, and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

2. Main Results

Theorem 2.1. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \cdots, T_N \colon K \to K$ be asymptotically quasi-nonexpansive mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^{\infty} r_n^i < \infty$, for all $1 \leq i \leq N$ and $F = \cap_{i=1}^{N} F(T_i) \neq \phi$. Let $\{\alpha_n^i\}$, $\{\beta_n^i\}$ and $\{\gamma_n^i\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, \cdots, N$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by (1), where $\{u_n^i\}$ are bounded sequences in K with $\sum_{n=1}^{\infty} u_n^i < \infty$. Then*

(i) *For* $x^* \in F$ *and for some sequences* $\{b_n^i\}$ *and* $\{d_n^i\}$ *for all* $i = 1, 2, \cdots, N$, $n \geq 1$, *of numbers satisfy* $\sum_{n=1}^{\infty} b_n^i < \infty$ *and* $\sum_{n=1}^{\infty} d_n^i < \infty$ *such that*

$$\|x_{n+1} - x^*\| = \|x_n^N - x^*\| \leq (1 + b_n^{N-1})\|x_n - x^*\| + d_n^{N-1},$$

(ii) *There exists a constant* $M > 0$ *such that*

$$\|x_{n+m} - x^*\| \leq M\|x_n - x^*\|$$

for all $n, m \geq 1$ *and* $x^* \in F$.

Proof. (i) Let $x^* \in F$. Then from (1) we have

$$\begin{aligned}
\|x_n^1 - x^*\| &= \|\alpha_n^1 T_1^n x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1\| \\
&\leq \alpha_n^1 \|T_1^n x_n - x^*\| + \beta_n^1 \|x_n - x^*\| + \gamma_n^1 \|u_n^1 - x^*\| \\
&\leq \alpha_n^1 (1 + r_n^1)\|x_n - x^*\| + \beta_n^1 \|x_n - x^*\| + \gamma_n^1 \|u_n^1 - x^*\| \\
&\leq (1 - \beta_n^1)(1 + r_n^1)\|x_n - x^*\| + \beta_n^1 (1 + r_n^1)\|x_n - x^*\| \\
&\quad + \gamma_n^1 \|u_n^1 - x^*\| \\
&\leq (1 + r_n^1)\|x_n - x^*\| + \gamma_n^1 \|u_n^1 - x^*\| \\
&\leq (1 + r_n^1)\|x_n - x^*\| + d_n^0,
\end{aligned}$$

where $d_n^0 = \gamma_n^1 \|u_n^1 - x^*\|$. Since $\sum_{n=1}^{\infty} \gamma_n^1 < \infty$, then $\sum_{n=1}^{\infty} d_n^0 < \infty$.

Next, we note that

$$\begin{aligned}
\|x_n^2 - x^*\| &= \|\alpha_n^2 T_2^n x_n^1 + \beta_n^2 x_n + \gamma_n^2 u_n^2\| \\
&\leq \alpha_n^2 \|T_2^n x_n^1 - x^*\| + \beta_n^2 \|x_n - x^*\| + \gamma_n^2 \|u_n^2 - x^*\| \\
&\leq \alpha_n^2 (1 + r_n^2)\|x_n^1 - x^*\| + \beta_n^2 \|x_n - x^*\| + \gamma_n^2 \|u_n^2 - x^*\| \\
&\leq \alpha_n^2 (1 + r_n^2)[(1 + r_n^1)\|x_n - x^*\| + d_n^0] + \beta_n^2 \|x_n - x^*\| \\
&\quad + \gamma_n^2 \|u_n^2 - x^*\| \\
&\leq [(1 + r_n^1)(1 + r_n^2)\alpha_n^2 + \beta_n^2]\|x_n - x^*\| + \alpha_n^2 (1 + r_n^2) d_n^0 \\
&\quad + \gamma_n^2 \|u_n^2 - x^*\| \\
&\leq (\alpha_n^2 + \beta_n^2)(1 + r_n^1)(1 + r_n^2)\|x_n - x^*\| + \alpha_n^2 (1 + r_n^2) d_n^0 \\
&\quad + \gamma_n^2 \|u_n^2 - x^*\| \\
&\leq (1 + r_n^1 + r_n^2 + r_n^1 r_n^2)\|x_n - x^*\| + \alpha_n^2 (1 + r_n^2) d_n^0 + \gamma_n^2 \|u_n^2 - x^*\| \\
&\leq (1 + b_n^1)\|x_n - x^*\| + d_n^1,
\end{aligned}$$

where $d_n^1 = \alpha_n^2 (1 + r_n^2) d_n^0 + \gamma_n^2 \|u_n^2 - x^*\|$ and $b_n^1 = (1 + r_n^1 + r_n^2 + r_n^1 r_n^2)$. Since $\sum_{n=1}^{\infty} d_n^0 < \infty$, $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$, $\sum_{n=1}^{\infty} r_n^i < \infty$ for $i = 1, 2$, and so $\sum_{n=1}^{\infty} d_n^1 < \infty$

and $\sum_{n=1}^{\infty} b_n^1 < \infty$. Similarly, we have

$$\begin{aligned}
\|x_n^3 - x^*\| &= \|\alpha_n^3 T_3^n x_n^2 + \beta_n^3 x_n + \gamma_n^3 u_n^3\| \\
&\le \alpha_n^3 \|T_3^n x_n^2 - x^*\| + \beta_n^3 \|x_n - x^*\| + \gamma_n^3 \|u_n^3 - x^*\| \\
&\le \alpha_n^3 (1 + r_n^3) \|x_n^2 - x^*\| + \beta_n^3 \|x_n - x^*\| + \gamma_n^3 \|u_n^3 - x^*\| \\
&\le \alpha_n^3 (1 + r_n^3)[(1 + b_n^1)\|x_n - x^*\| + d_n^1] + \beta_n^3 \|x_n - x^*\| \\
&\quad + \gamma_n^3 \|u_n^3 - x^*\| \\
&\le [\alpha_n^3 (1 + r_n^3)(1 + b_n^1) + \beta_n^3] \|x_n - x^*\| + \alpha_n^3 (1 + r_n^3) d_n^1 \\
&\quad + \gamma_n^3 \|u_n^3 - x^*\| \\
&\le (\alpha_n^3 + \beta_n^3)(1 + b_n^1)(1 + r_n^3) \|x_n - x^*\| + \alpha_n^3 (1 + r_n^3) d_n^1 \\
&\quad + \gamma_n^3 \|u_n^3 - x^*\| \\
&\le (1 + b_n^1)(1 + r_n^3) \|x_n - x^*\| + d_n^2 \\
&\le (1 + b_n^2) \|x_n - x^*\| + d_n^2,
\end{aligned}$$

where $b_n^2 = b_n^1 + r_n^3 + b_n^1 r_n^3$ and $d_n^2 = \alpha_n^3 (1 + r_n^3) d_n^1 + \gamma_n^3 \|u_n^3 - x^*\|$. Since $\sum_{n=1}^{\infty} b_n^1 < \infty$, $\sum_{n=1}^{\infty} r_n^3 < \infty$, $\sum_{n=1}^{\infty} d_n^1 < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^3 < \infty$, so $\sum_{n=1}^{\infty} b_n^2 < \infty$ and $\sum_{n=1}^{\infty} d_n^2 < \infty$.

By continuing the above process, there exist nondecreasing sequences $\{d_n^{l-1}\}$ and $\{b_n^{l-1}\}$ such that $\sum_{n=1}^{\infty} d_n^{l-1} < \infty$ and $\sum_{n=1}^{\infty} b_n^{l-1} < \infty$ and

$$\|x_n^i - x^*\| \le (1 + b_n^{i-1}) \|x_n - x^*\| + d_n^{i-1}, \quad \forall n \ge 1, \quad \forall i = 1, 2, \cdots, N.$$

Thus

$$\|x_{n+1} - x^*\| = \|x_n^N - x^*\| \le (1 + b_n^{N-1}) \|x_n - x^*\| + d_n^{N-1}, \quad \forall n \in N.$$

This completes the proof of (i).

(ii) Since $1 + x \le e^x$ for all $x > 0$. Then from (i) it can be obtained that

$$\begin{aligned}
&\|x_{n+m} - x^*\| \\
&\le (1 + b_{n+m-1}^{N-1}) \|x_{n+m-1} - x^*\| + d_{n+m-1}^{N-1} \\
&\le e^{b_{n+m-1}^{N-1}} \|x_{n+m-1} - x^*\| + d_{n+m-1}^{N-1} \\
&= e^{(b_{n+m-1}^{N-1} + b_{n+m-2}^{N-1})} \|x_{n+m-2} - x^*\| + e^{b_{n+m-1}^{N-1}} d_{n+m-2}^{N-1} + d_{n+m-1}^{N-1} \\
&= e^{(b_{n+m-1}^{N-1} + b_{n+m-2}^{N-1})} \|x_{n+m-2} - x^*\| + e^{b_{n+m-1}^{N-1}} (d_{n+m-1}^{N-1} + d_{n+m-2}^{N-1}) \\
&\cdots \\
&= e^{\sum_{k=n}^{n+m-1} b_k^{N-1}} \|x_n - x^*\| + e^{\sum_{k=n}^{n+m-1} b_k^{N-1}} . \sum_{k=n}^{n+m-1} d_k^{N-1}
\end{aligned}$$

$$= M.\|x_n - x^*\| + M.\sum_{k=n}^{n+m-1} d_k^{N-1},$$

where $M = e^{\sum_{k=n}^{\infty} b_k^{N-1}}$. This completes the proof of (ii). □

Theorem 2.2. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \cdots, T_N : K \to K$ be N asymptotically quasi-nonexpansive mappings ($\{T_i : i = 1, 2, \cdots, N\}$ need not be continuous) with sequences $\{r_n^i\}$ such that $\sum_{n=1}^{\infty} r_n^i < \infty$, for all $1 \le i \le N$ and $F = \cap_{i=1}^{N} F(T_i) \ne \phi$. Let $\{\alpha_n^i\}$, $\{\beta_n^i\}$ and $\{\gamma_n^i\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, \cdots, N$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by (1), where $\{u_n^i\}$ are bounded sequences in K with $\sum_{n=1}^{\infty} u_n^i < \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \cdots, T_N\}$ if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0,$$

where $d(y, A)$ denotes the distance of y to the set A; that is, $d(y, A) = \inf\{\|y - z\| : \forall z \in A\}$.

Proof. Suppose $\{x_n\}$ converges strongly to a common fixed point z of $\{T_1, T_2, \cdots, T_N\}$. Then

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

Conversely, suppose $\{T_1, T_2, \cdots, T_N\}$. Then from Theorem 2.1 (i), we have

$$\|x_{n+1} - x^*\| \le (1 + b_n^{N-1})\|x_n - x^*\| + d_n^{N-1}, \quad \forall n \in N, \quad \forall x^* \in F. \tag{2}$$

Since $\sum_{n=1}^{\infty} u_n^i < \infty$, $\sum_{n=1}^{\infty} r_n^i < \infty$ for all $1 \le i \le N$, we know that $\sum_{n=1}^{\infty} b_n^{N-1} < \infty$ and $\sum_{n=1}^{\infty} d_n^{N-1} < \infty$. So from (2), we obtain

$$d(x_{n+1}, F) \le (1 + b_n^{N-1}) d(x_n, F) + d_n^{N-1}.$$

Since $\liminf_{n\to\infty} d(x_n, F) = 0$ and from Lemma 2.1, we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Next we will show that $\{x_n\}$ is a Cauchy sequence. For all $\varepsilon > 0$, from Theorem 2.1 (ii), it can be known that there must exists a constant $M > 0$, such that

$$\|x_{n+m} - x^*\| \le M.\|x_n - x^*\| + M.\sum_{k=n}^{n+m-1} d_k^{N-1}, \ \forall m, n \in N, \ \forall x^* \in F. \tag{3}$$

Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} d_n^{N-1} < \infty$, there must exists a constant N_1, such that when $n \ge N_1$

$$d(x_n, F) < \frac{\varepsilon_1}{3M} \quad \text{and} \quad \sum_{k=n}^{n+m-1} d_k^{N-1} < \frac{\varepsilon_1}{6M}.$$

So there must exists $w^* \in F$, such that

$$d(x_{N_1}, w^*) = \|x_{N_1} - w^*\| < \frac{\varepsilon_1}{3M}.$$

From (3), it can be obtained that when $n \geq N_1$

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - w^*\| + \|x_n - w^*\| \\
&\leq M\|x_{N_1} - w^*\| + M\|x_{N_1} - w^*\| + 2M \sum_{k=N_1}^{\infty} d_n^{N-1} \\
&\leq M.\frac{\varepsilon_1}{3M} + M.\frac{\varepsilon_1}{3M} + 2M.\frac{\varepsilon_1}{6M} \\
&< \varepsilon_1,
\end{aligned}$$

that is,

$$\|x_{n+m} - x_n\| < \varepsilon_1.$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent from the completeness of E. Let $\lim_{n\to\infty} x_n = y^*$. Then $y^* \in K$. It remains to show that $y^* \in F$. Let $\varepsilon_2 > 0$ be given. Then there exists a natural number N_2 such that

$$\|x_n - y^*\| < \frac{\varepsilon_2}{2(2 + r_1^i)}, \quad \forall n \geq N_2.$$

Since $\lim_{n\to\infty} d(x_n, F) = 0$, there must exists a natural number $N_3 \geq N_2$ such that for all $n \geq N_3$, we have

$$d(x_n, F) < \frac{\varepsilon_2}{3(2 + r_1^i)}$$

and in particular we have

$$d(x_{N_3}, F) < \frac{\varepsilon_2}{3(2 + r_1^i)}.$$

Therefore, there exists $z^* \in F$ such that

$$\|x_{N_3} - z^*\| < \frac{\varepsilon_2}{2(2 + r_1^i)}.$$

Consequently, we have

$$\begin{aligned}
\|T_iy^* - y^*\| &= \|T_iy^* - z^* + z^* - x_{N_3} + x_{N_3} - y^*\| \\
&\le \|T_iy^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le (1 + r_1^i)\|y^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le (1 + r_1^i)[\|y^* - x_{N_3}\| + \|x_{N_3} - z^*\|] + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le (2 + r_1^i)\|y^* - x_{N_3}\| + (2 + r_1^i)\|x_{N_3} - z^*\| \\
&< (2 + r_1^i).\frac{\varepsilon_2}{2(2 + r_1^i)} + (2 + r_1^i).\frac{\varepsilon_2}{2(2 + r_1^i)} \\
&< \varepsilon_2.
\end{aligned}$$

This implies that $y^* \in F(T_i)$ for all $i = 1, 2, \cdots, N$. Hence we conclude that $y^* \in F = \cap_{i=1}^N F(T_i)$, that is, y^* is a common fixed point of the mappings $\{T_1, T_2, \cdots, T_N\}$. Thus $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \cdots, T_N\}$. This completes the proof. □

Remark 2.3. Theorem 2.2 extends Theorem 3.2 of Shahzad and Udomene [14] to the case of multistep iterative sequences with errors for finite family of asymptotically quasi-nonexpansive mappings.

Remark 2.4. Theorem 2.2 also extends and improves Theorem 1.1 and Theorem 1.1' of Petryshyn and Williamson [8] and Theorem 3.1 of Ghosh and Debnath [4] to the case of multistep iterative sequences with errors for finite family of more general class of quasi-nonexpansive mappings. The continuity of mapping is relaxed in Theorem 2.2 as compared to [8, 4].

Remark 2.5. Theorem 2.2 also extends Liu [9, Theorem 1] and Liu [10, Theorem 1] to the case of multistep iterative sequences with errors for finite family of asymptotically quasi-nonexpansive mappings.

Theorem 2.6. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \cdots, T_N : K \to K$ be N uniformly continuous asymptotically quasi-nonexpansive mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^\infty r_n^i < \infty$, for all $1 \le i \le N$ and $F = \cap_{i=1}^N F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence defined by (1) with $\sum_{n=1}^\infty \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1 - \varepsilon]$ for all $i = 1, 2, \cdots, N$, for some $\varepsilon \in (0, 1)$. Then*

$$\|x_n - T_i^n x_n\| = 0$$

for all $i = 1, 2, \cdots, N$.

Proof. Let $x^* \in F = \cap_{i=1}^N F(T_i)$. Then by Theorem 2.1 (i) and Lemma 1.1, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Let $\lim_{n\to\infty} \|x_n - x^*\| = a$. If $a = 0$, then by the continuity of each T_i the conclusion follows. Now suppose that $a > 0$. Firstly, we are now to show that $\lim_{n\to\infty}\|T_N^n x_n - x_n\| = 0$. Since $\{x_n\}$ and $\{u_n^i\}$ are bounded for all $i = 1, 2, \cdots, N$, there exists $R > 0$ such that

$$x_n - x^* + \gamma_n^i(u_n^i - x_n), T_i^n x_n^{i-1} - x^* + \gamma_n^i(u_n^i - x_n) \in B_R(0)$$

for all $n \geq 1$ and for all $i = 1, 2, \cdots, N$. Using Lemma 1.2, we have

$$\begin{aligned}
&\|x_n^N - x^*\|^2 \\
&= \|\alpha_n^N T_N^n x_n^{N-1} + \beta_n^N x_n + \gamma_n^N u_n^N - x^*\|^2 \\
&= \|\alpha_n^N (T_N^n x_n^{N-1} - x^* + \gamma_n^N(u_n^N - x_n)) + (1-\alpha_n^N)(x_n - x^* + \gamma_n^N(u_n^N - x_n))\|^2 \\
&\leq \alpha_n^N \|T_N^n x_n^{N-1} - x^* + \gamma_n^N(u_n^N - x_n)\|^2 \\
&\quad + (1-\alpha_n^N)\|x_n - x^* + \gamma_n^N(u_n^N - x_n)\|^2 - W_2(\alpha_n^N) g(\|T_N^n x_n^{N-1} - x_n)\| \\
&\leq \alpha_n^N (\|T_N^n x_n^{N-1} - x^*\| + \gamma_n^N \|u_n^N - x_n)\|^2 \\
&\quad + (1-\alpha_n^N)(\|x_n - x^*\| + \gamma_n^N\|u_n^N - x_n\|)^2 - W_2(\alpha_n^N) g(\|T_N^n x_n^{N-1} - x_n\|) \\
&\leq \alpha_n^N [(1+b_n^{N-2})\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N\|u_n^N - x_n\|]^2 \\
&\quad + (1-\alpha_n^N)[(1+b_n^{N-2})\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N\|u_n^N - x_n\|]^2 \\
&\quad - W_2(\alpha_n^N) g(\|T_N^n x_n^{N-1} - x_n\|) \\
&\leq [(1+b_n^{N-2})\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N\|u_n^N - x_n\|]^2 \\
&\quad - W_2(\alpha_n^N) g(\|T_N^n x_n^{N-1} - x_n\|) \\
&\leq [\|x_n - x^*\| + \lambda_n^{N-2}]^2 - W_2(\alpha_n^N) g(\|T_N^n x_n^{N-1} - x_n\|),
\end{aligned} \tag{4}$$

where $\lambda_n^{N-2} = d_n^{N-2} + \gamma_n^N\|u_n^N - x_n\|$. Observe that $\varepsilon^3 \leq W_2(\alpha_n^N)$, it implies that

$$\varepsilon^3 g(\|T_N^n x_n^{N-1} - x_n\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \rho_n^{N-2},$$

where

$$\rho_n^{N-2} = 2\lambda_n^{N-2} + (\lambda_n^{N-2})^2.$$

Since $\sum_{n=1}^\infty d_n^{N-2} < \infty$ and $\sum_{n=1}^\infty \gamma_n^{N-2} < \infty$, we get $\sum_{n=1}^\infty \rho_n^{N-2} < \infty$. This implies that

$$\lim_{n\to\infty} g(\|T_N^n x_n^{N-1} - x_n\|) = 0.$$

Since g is strictly increasing and continuous at 0, it follows that

$$\lim_{n\to\infty} \|T_N^n x_n^{N-1} - x_n\| = 0.$$

Since $T_N, \forall N$ is asymptotically quasi-nonexpansive, note that

$$\begin{aligned}
\|x_n - x^*\| &\leq \|x_n - T_N^n x_n^{N-1}\| + \|T_N^n x_n^{N-1} - x^*\| \\
&= \|x_n - T_N^n x_n^{N-1}\| + (1 + r_n^N)\|x_n - x^*\|
\end{aligned}$$

for all $n \geq 1$. Thus

$$
\begin{aligned}
a &= \lim_{n\to\infty} \|x_n - x^*\| \\
&\le \liminf_{n\to\infty} \|x_n^{N-1} - x^*\| \\
&\le \limsup_{n\to\infty} \|x_n^{N-1} - x^*\| \\
&\le a
\end{aligned}
$$

and therefore $\lim_{n\to\infty} \|x_n^{N-1} - x^*\| = a$. Using the same argument in the proof above, we have

$$
\begin{aligned}
&\|x_n^{N-1} - x^*\|^2 \\
&\le \alpha_n^{N-1}\|T_N^n x_n^{N-2} - x^* + \gamma_n^{N-1}(u_n^{N-1} - x_n)\|^2 \\
&\quad + (1-\alpha_n^{N-1})\|x_n - x^* + \gamma_n^{N-1}(u_n^{N-1} - x_n)\|^2 \\
&\quad - W_2(\alpha_n^{N-1})g(\|T_{N-1}^n x_n^{N-2} - x_n\|) \\
&\le \alpha_n^{N-1}[(1+b_n^{N-3})\|x_n - x^*\| + d_n^{N-3} + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|]^2 \\
&\quad + (1-\alpha_n^{N-1})[(1+b_n^{N-3})\|x_n - x^*\| + d_n^{N-3} + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|]^2 \\
&\quad - W_2(\alpha_n^{N-1})g(\|T_{N-1}^n x_n^{N-2} - x_n\|) \\
&\le [(1+b_n^{N-3})\|x_n - x^*\| + d_n^{N-3} + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|]^2 \\
&\quad - W_2(\alpha_n^{N-1})g(\|T_{N-1}^n x_n^{N-2} - x_n\|) \\
&\le [\|x_n - x^*\| + \lambda_n^{N-3}]^2 - W_2(\alpha_n^{N-1})g(\|T_{N-1}^n x_n^{N-2} - x_n\|),
\end{aligned}
\tag{5}
$$

where $\lambda_n^{N-3} = d_n^{N-3} + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|$. This implies that

$$
\varepsilon^3 g(\|T_{N-1}^n x_n^{N-2} - x_n\|) \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \rho_n^{N-3},
$$

where $\rho_n^{N-3} = 2\lambda_n^{N-3} + (\lambda_n^{N-3})^2$ and therefore

$$
\lim_{n\to\infty} \|T_{N-1}^n x_n^{N-2} - x_n\| = 0.
$$

Thus, we have

$$
\begin{aligned}
&\|x_n - T_N^n x_n\| \\
&\le \|x_n - T_N^n x_n^{N-1}\| + \|T_N^n x_n^{N-1} - T_N^n x_n\| \\
&\le \|x_n - T_N^n x_n^{N-1}\| + (1+r_n^N)\|x_n^{N-1} - x_n\| \\
&\le \|x_n - T_N^n x_n^{N-1}\| + (1+r_n^N)\|\alpha_n^{N-1}T_{N-1}^n x_n^{N-2} + \beta_n^{N-1}x_n + \gamma_n^{N-1}u_n^{N-1} - x_n\| \\
&\le \|x_n - T_N^n x_n^{N-1}\| + (1+r_n^N)[\alpha_n^{N-1}\|T_{N-1}^n x_n^{N-2} - x_n\| + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|].
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_n - T_N^n x_n^{N-1}\| = 0$, $\lim_{n\to\infty}\|x_n - T_{N-1}^n x_n^{N-2}\| = 0$, and $\sum_{n=1}^{\infty}\gamma_n^{N-1} < \infty$, $\sum_{n=1}^{\infty} r_n^N < \infty$, it follows that

$$\lim_{n\to\infty}\|x_n - T_N^n x_n\| = 0.$$

Similarly, by using the same argument as in the proof above we have $\lim_{n\to\infty}\|x_n - T_{N-2}^n x_n^{N-3}\| = \lim_{n\to\infty}\|x_n - T_{N-3}^n x_n^{N-4}\| =, \cdots, = \lim_{n\to\infty}\|x_n - T_2^n x_n^1\| = 0$. This implies that

$$\begin{aligned}\lim_{n\to\infty}\|x_n - T_{N-1}^n x_n\| &= \lim_{n\to\infty}\|x_n - T_{N-2}^n x_n\|\\ &\cdots\\ &= \lim_{n\to\infty}\|x_n - T_3^n x_n\| = 0.\end{aligned}$$

It remains to show that

$$\lim_{n\to\infty}\|x_n - T_1^n x_n\| = 0, \qquad \lim_{n\to\infty}\|x_n - T_2^n x_n\| = 0.$$

Note that

$$\begin{aligned}&\|x_n^1 - x^*\|^2\\ &\leq \alpha_n^1(\|T_1^n x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*)\|^2 + (1-\alpha_n^1)(\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*)\|^2\\ &\quad - W_2(\alpha_n^1)g(\|T_1^n x_n - x_n\|)\\ &\leq \alpha_n^1[(1+r_n^1)\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\|]^2\\ &\quad + (1-\alpha_n^1)[(1+r_n^1)\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\|]^2 - W_2(\alpha_n^1)g(\|T_1^n x_n - x_n\|)\\ &\leq [(1+r_n^1)\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\|]^2 - W_2(\alpha_n^1)g(\|T_1^n x_n - x_n\|)\\ &\leq [\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\|]^2 - W_2(\alpha_n^1)g(\|T_1^n x_n - x_n\|)\end{aligned}$$

Thus, we have

$$\varepsilon^3 g(\|T_1^n x_n - x_n\|) \leq [\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\|]^2 - \|x_n^1 - x^*\|^2$$

and therefore

$$\lim_{n\to\infty}\|T_1^n x_n - x_n\| = 0.$$

Since

$$\begin{aligned}\|x_n - T_2^n x_n\| &\leq \|x_n - T_2^n x_n^1\| + \|T_2^n x_n^1 - T_2^n x_n\|\\ &\leq \|x_n - T_2^n x_n^1\| + (1+r_n^2)\|x_n^1 - x_n\|\\ &\leq \|x_n - T_2^n x_n^1\| + (1+r_n^2)\|\alpha_n^1 T_1^n x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1 - x_n\|\\ &\leq \|x_n - T_2^n x_n^1\| + (1+r_n^2)[\alpha_n^1\|T_1^n x_n - x_n\| + \gamma_n^1\|u_n^1 - x_n\|],\end{aligned}$$

it implies that $\lim_{n\to\infty}\|T_2^n x_n - x_n\| = 0$. Therefore $\lim_{n\to\infty}\|T_i^n x_n - x_n\| = 0$ for all $i = 1, 2, \cdots, N$. $\square$

Theorem 2.7. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \cdots, T_N \colon K \to K$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^{\infty} r_n^i < \infty$, for all $1 \le i \le N$ and $F = \cap_{i=1}^N F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence defined by (1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \cdots, N$, for some $\varepsilon \in (0,1)$. Then*

$$\|x_n - T_i x_n\| = 0$$

for all $i = 1, 2, \cdots, N$.

Proof. It follows from Theorem 2.6, that

$$\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0$$

for all $i = 1, 2, \cdots, N$ and this implies that

$$\|x_{n+1} - x_n\| \le \alpha_n^N \|T_N^n x_n^{N-1} - x_n\| + \gamma_n^N \|u_n^N - x_n\| \to 0,$$

as $n \to \infty$. We now have to show that $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \cdots, N$. Observe that

$$\begin{aligned}
\|x_n - T_1^{n-1} x_n\| \le {} & \|x_n - x_{n-1}\| + \|x_{n-1} - T_1^{n-1} x_{n-1}\| \\
& + \|T_1^{n-1} x_{n-1} - T_1^{n-1} x_n\| \\
\le {} & \|x_n - x_{n-1}\| + \|x_{n-1} - T_1^{n-1} x_{n-1}\| \\
& + L\|x_{n-1} - x_n\| \to 0,
\end{aligned}$$

as $n \to \infty$. Using the above inequality, we have

$$\begin{aligned}
\|x_n - T_1 x_n\| &\le \|x_n - T_1^n x_n\| + \|T_1^n x_n - T_1 x_n\| \\
&\le \|x_n - T_1^n x_n\| + L\|T_1^{n-1} x_n - x_n\| \to 0,
\end{aligned}$$

as $n \to \infty$. Again we have

$$\begin{aligned}
&\|x_n - T_2^{n-1} x_n^1\| \\
&\le \|x_n - x_{n-1}\| + \|x_{n-1} - T_2^{n-1} x_{n-1}^1\| + \|T_2^{n-1} x_{n-1}^1 - T_2^{n-1} x_n^1\| \\
&\le \|x_n - x_{n-1}\| + \|x_{n-1} - T_2^{n-1} x_{n-1}^1\| + L\|x_{n-1}^1 - x_n^1\| \\
&\le \|x_n - x_{n-1}\| + \|x_{n-1} - T_2^{n-1} x_{n-1}^1\| + L[\|x_{n-1}^1 - x_{n-1}\| \\
&\quad + \|x_{n-1} - x_n\| + \|x_n - x_n^1\|] \\
&\le (L+1)\|x_n - x_{n-1}\| + \|x_{n-1} - T_2^{n-1} x_{n-1}^1\| + L\|x_{n-1}^1 - x_{n-1}\| \\
&\quad + L\|x_n - x_n^1\| \to 0,
\end{aligned}$$

as $n \to \infty$. Using the above inequality, we have

$$\begin{aligned}
\|x_n - T_2 x_n\| &\le \|x_n - T_2^n x_n^1\| + \|T_2^n x_n^1 - T_2 x_n\| \\
&\le \|x_n - T_2^n x_n^1\| + L\|T_2^{n-1} x_n^1 - x_n\| \to 0,
\end{aligned}$$

as $n \to \infty$. Similarly, we can prove that

$$\|x_n - T_3x_n\| = 0, \|x_n - T_4x_n\| = 0, \cdots, \|x_n - T_Nx_n\| = 0.$$

Thus

$$\|x_n - T_ix_n\| = 0$$

for all $i = 1, 2, \cdots, N$. This completes the proof. □

Theorem 2.8. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \cdots, T_N \colon K \to K$ be N uniformly continuous asymptotically quasi-nonexpansive mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^{\infty} r_n^i < \infty$, for all $1 \leq i \leq N$ and $F = \cap_{i=1}^N F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence defined by (1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \cdots, N$, for some $\varepsilon \in (0, 1)$. Suppose T_1^m is compact for some $m \in N$. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \cdots, T_N\}$.*

Proof. Since T_1^m is compact for some $m \in N$, from the uniform continuity of T_1 and $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0$, we can show that for each $l \geq 1$,

$$\lim_{n\to\infty} \|T_1^l x_n - x_n\| = 0.$$

Since $\{x_n\}$ is bounded and T_1^m is compact, $\{T_1^m x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{T_1^m x_{n_j}\}_{j=1}^{\infty}$. Suppose $\lim_{j\to\infty} T_1^m x_{n_j} = x^* \in K$. Then the inequality

$$\|x_{n_j} - x^*\| \leq \|x_{n_j} - T_1^m x_{n_j}\| + \|T_1^m x_{n_j} - x^*\|$$

yields $\lim_{j\to\infty} \|x_{n_j} - x^*\| = 0$ which implies by Lemma 1.1 that $x_n \to x^* \in F = \cap_{i=1}^N F(T_i)$, since $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Thus $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \cdots, T_N\}$. This completes the proof. □

Corollary 2.9. *Let E be a real uniformly convex Banach space and K be a nonempty compact convex subset of E. Let $T_1, T_2, \cdots, T_N \colon K \to K$ be N continuous asymptotically quasi-nonexpansive mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^{\infty} r_n^i < \infty$, for all $1 \leq i \leq N$ and $F = \cap_{i=1}^N F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence defined by (1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \cdots, N$, for some $\varepsilon \in (0, 1)$. Suppose T_1^m is compact for some $m \in N$. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \cdots, T_N\}$.*

Remark 2.10. (1) Corollary 2.9 extends Theorem 1.2 of Khan and Takahashi [6] to the case of multistep iterative sequences with errors for finite family of more general class of mappings.

(2) Theorem 2.8 extends Theorem 2.2 of Schu [13] to the case of multistep iterative sequences with errors for finite family of more general class of mappings.

(3) Corollary 2.9 also extends the results of Liu [11] to the case of multistep iterative sequences for finite family of more general class of mappings.

For our next result, we shall need the following definition:

Definition 2.11. Let C be a nonempty closed subset of a Banach space E. A mapping $T\colon C \to C$ is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\lim_{i\to\infty} x_{n_i} = x \in C$.

Theorem 2.12. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \cdots, T_N\colon K \to K$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^{\infty} r_n^i < \infty$, for all $1 \le i \le N$ and $F = \cap_{i=1}^{N} F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence defined by (1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \cdots, N$, for some $\varepsilon \in (0,1)$. Suppose one of the mappings in $\{T_1, T_2, \cdots, T_N\}$ is semi-compact. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \cdots, T_N\}$.*

Proof. Suppose that T_1 is semi-compact. By Theorem 23.7, we have

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0. \tag{6}$$

So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = x^* \in K$. So from (6) we have $\lim_{n_j\to\infty} \|x_{n_j} - T_j x_{n_j}\| = 0$ for all $j = 1, 2, \cdots, N$ and so $\|x^* - T_j x^*\| = 0$ for all $j = 1, 2, \cdots, N$. This implies that $x^* \in F = \cap_{i=1}^{N} F(T_i)$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, it follows, as in the proof of Theorem 3.2, that $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \cdots, T_N\}$. This completes the proof. ☐

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