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LINEAR OPERATORS IN PROBABILISTIC NORMED SPACES

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Abstract. Probabilistic normed spaces had been redefined by Alsina, Schweizer and Sklar. In this paper, the boundedness notions for linear operators in probabilistic normed space was studied and the relation between operator and operation was discussed.

1. INTRODUCTION

Probabilistic normed spaces (PN spaces) were introduced by Šerstnev by means of a definition that was closely modeled on the theory of normed spaces. Here we consistantly adopt the new, and in our opinion convincing, definition of PN space given in the paper by Alsina, Schweizer and Sklar[1]. The notation and concepts used are those of [3,5,and 6]. In [2], it discussed the operator, for example, strongly bounded operator, certainly bounded operator, perhaps bounded operator and it also proved the relation between them. And from this paper, we know in the class of linear operators, no two of the concepts of certain boundedness and continuity imply each other.

In the following, the letter Δ^+ will be denoted the set of one-dimensional distribution functions which is nondecreasing, left-continuous and v(0) = 0, and has both a maximal element ϵ_0 and a minimal element ϵ_{∞} : these are given

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Changli Yan

respectively, by

$$\epsilon_0(x) = \begin{cases} 0, & x \le 0\\ 1, & x > 0 \end{cases}$$
(1.1)

and

$$\epsilon_{\infty}(x) = \begin{cases} 0, & x < \infty \\ 1, & x = \infty. \end{cases}$$
(1.2)

We shall also consider the subset $D^+ \subset \Delta^+$ of the proper distance distribution function, i.e., those $F \in \Delta^+$ for which $\lim_{x\to\infty} F(x) = 1$.

Definition 1.1. [1] A probabilistic normed space (briefly, PN space) is a quadruple (V, v, τ, τ^*) , where V is a real vector space, τ and τ^* are continuous triangle functions with $\tau \leq \tau^*$ and v is a mapping from V into Δ^+ such that for all p,q in V, the following conditions hold:

(PN1) $v_p = \varepsilon_0$ if and only if $p=\theta$; (PN2) $v_{-p} = v_p$; (PN3) $v_{p+q} \ge \tau(v_p, v_q)$;

(**PN4**) $v_p \leq \tau^*(v_{\alpha p}, v_{(1-\alpha)p})$ for all α in [0,1], for the continuous t-norm T such that $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, where

$$T^*(x,y) = 1 - T(1 - x, 1 - y),$$

$$\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s), G(t))$$

and

$$\tau_{T^*}(F,G)(x) = \inf_{s+t=x} T^*(F(s),G(t)).$$

A PN space is called a Šerstnev space if it satisfies (PN1), (PN3) and the following Šerstnev condition which implies both (PN2) and (PN4).

$$N_{ap}(x) = N_p(\frac{x}{a})$$

for any $p \in S, a \in R/\{0\}, x > 0$.

There is a natural topology in a PN space (V, v, τ, τ^*) which is called the strong topology. It is defined by the neighbourhood

$$N_p(t) = \{q \in V : v_{q-p}(t) > 1 - t\} = \{q \in V : d_l(v_{q-p}, \epsilon_0) < t\},\$$

where t > 0, d_l is the modified Levy metric(see [2]). For every t > 0, the neighbourhood $N_{\theta}(t)$ at θ of V is defined by

$$N_{\theta}(t) = \{ p | p \in V, v_p(t) > 1 - t \} = \{ p | p \in V, d_l(v_p, \epsilon_0) < t \}$$

Definition 1.2. [3] Let (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be PN spaces. A linear map $T: V \to V'$ is said to be:

(i) Certainly bounded if every certainly bounded set A of the space (V, v, τ, τ^*) has, as image by T a certainly bounded set TA of the space $(V', \mu, \sigma, \sigma^*)$, i.e., if there exists $x_0 \in (0, +\infty)$ such that $v_p(x_0) = 1$ for all $p \in A$, then there exists $x_1 \in (0, +\infty)$ such that $\mu_{Tp}(x_1) = 1$ for all $p \in A$.

(ii) Bounded if it map every D-bounded set of V into a D-bounded set of V', i.e., it satisfies the implication,

$$\lim_{x \to +\infty} \varphi_A(x) = 1 \Rightarrow \lim_{x \to +\infty} \varphi_{TA}(x) = 1$$

for every nonempty subset A of V.

(iii) Strongly B-bounded if there exists a constant k > 0 such that, for every $p \in V$ and for every x > 0, $\mu_{Tp}(x) \ge v_p(\frac{x}{k})$, or equivalently if there exists a constant h > 0 such that, for every $p \in V$ and for every x > 0,

$$\mu_{Tp}(hx) \ge v_p(x).$$

(iv) Strongly C-bounded if there exists a constant $h \in (0, 1)$ such that for every $p \in V$ and for every x > 0,

$$v_p(x) > 1 - x \Rightarrow \mu_{Tp}(hx) > 1 - hx.$$

Definition 1.3. Let (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be PN spaces. A linear map $T: V \to V'$ is said to be infer-strongly B-bounded if there exists M > 0, for any $p \in V, x > 0, y > 0$ such that

$$\mu_{Tp}(Mx + My) \ge \tau(v_{\frac{p}{2}}(x), v_{\frac{p}{2}}(y)).$$

Definition 1.4. A linear map $T : (V, v, \tau, \tau^*) \to (V', \mu, \sigma, \sigma^*)$ is said to be semi-bounded if it maps every D-bounded set of V into a semi-bounded set of V', i.e., it satisfies the implication

$$\lim_{x \to +\infty} \phi_A(x) = 1 \Rightarrow \lim_{x \to +\infty} \phi_{TA}(x) < 1,$$

for every nonempty subset A of V.

2. Main Result

Theorem 2.1. Let (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be PN spaces. Let $T : V \to V'$ be a linear operator.

i) If T is strongly B-bounded, then it is infer-strongly B-bounded.

ii) If T is infer-strongly B-bounded, then it is continuous.

Proof. i) By definition, there exists a constant M > 0 such that for every $p \in V$ and for every x, y > 0,

 $\mu_{Tp}(Mx + My) \ge v_p(x+y) \ge \tau(v_{\frac{p}{2}}, v_{\frac{p}{2}})(x+y) \ge \tau(v_{\frac{p}{2}}(x), v_{\frac{p}{2}}(y)).$ Therefore, i) holds.

ii) If $p_n \to p$, then $\mu_{Tp_n-Tp}(Mx) = \mu_{T(p_n-p)}(Mx) \ge \tau(v_{\frac{p_n-p}{2}}(\frac{x}{2}), v_{\frac{p_n-p}{2}}(\frac{x}{2}))$, and $\tau(v_{\frac{p_n-p}{2}}(\frac{x}{2}), v_{\frac{p_n-p}{2}}(\frac{x}{2})) \to \tau(\epsilon_0, \epsilon_0)(\frac{x}{2}) = 1$ as $n \to \infty$. Hence, we have $Tp_n - Tp \to \theta$, i.e., $Tp_n \to Tp$, for any x > 0, M > 0. Therefore, T is continuous. \Box

But the converse need not to be true.

Example 2.1. Let V be a vector space and let $v_{\theta} = \mu_{\theta} = \varepsilon_0$. For $p, q \neq \theta$ and $x \in R$, if

$$\epsilon_p(x) = \begin{cases} 0, & x \le 1\\ 1, & x > 1, \end{cases}$$
(2.1)

$$\mu_p(x) = \begin{cases} 0, & x \le 0\\ \frac{1}{3}, & 0 < x \le 1\\ \frac{9}{10}, & 1 < x < \infty\\ 1, & x = \infty, \end{cases}$$
(2.2)

and

$$\tau(v_p(x), v_q(y)) = \tau^*(v_p(x), v_q(y)) = \min(v_p(x), v_q(x)), \\ \delta(\mu_p(x), \mu_q(y)) = \delta^*(\mu_p(x), \mu_q(y)) = \min(\mu_p(x), \mu_q(x))$$

 $o(\mu_p(x), \mu_q(y)) = o^*(\mu_p(x), \mu_q(y)) = \min(\mu_p(x), \mu_q(x)),$ then (V, v, τ, τ^*) and $(V, \mu, \delta, \delta^*)$ are equilateral PN spaces. Now let I: $(V, v, \tau, \tau^*) \to (V, \mu, \delta, \delta^*)$ be the identity operator. Then I is not strongly B-bounded, because for every k > 0 and for $x = \max\{2, \frac{1}{k}\},$

$$u_{Ip}(kx) = \frac{9}{10} < 1 = v_p(x).$$

But I is infer-strongly B-bounded, because for M > 0 and for all $x \leq y$,

$$p(Mx + My) \ge \delta(\mu_{\frac{p}{2}}(Mx), \mu_{\frac{p}{2}}(My)) = \delta^{*}(\mu_{\frac{p}{2}}(Mx), \mu_{\frac{p}{2}}(My)) = \min(\mu_{\frac{p}{2}}(Mx), \mu_{\frac{p}{2}}(Mx)) = \mu_{\underline{p}}(Mx)$$

and $\tau(v_{\frac{p}{2}}(x), v_{\frac{p}{2}}(y)) = \tau^*(v_{\frac{p}{2}}(x), v_{\frac{p}{2}}(y)) = \min(v_{\frac{p}{2}}(x), v_{\frac{p}{2}}(x)) = v_{\frac{p}{2}}(x)$, there exists $M = \frac{1}{2} > 0$, for $x = \frac{1}{2} \in R$, $\mu_{\frac{p}{2}}(Mx) = \mu_{\frac{p}{2}}(\frac{1}{2}x) = \frac{1}{3}$, $v_{\frac{p}{2}}(x) = 0$. Therefore, we obtain

$$\mu_{Ip}(Mx + My) \ge \mu_{\frac{p}{2}}(Mx) > \tau(v_{\frac{p}{2}}(x), v_{\frac{p}{2}}(y)) = v_{\frac{p}{2}}(x)$$

Theorem 2.2. (see [2]) Let (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be PN spaces. A linear map $T: V \to V'$ is either continuous at every point of V or at no point of V.

Theorem 2.3. (see [2]) Let (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be two PN spaces and let $T : (V, v, \tau, \tau^*) \rightarrow (V', \mu, \sigma, \sigma^*)$ be a linear map. If there exists a constant h > 0 such that, for every x > 0 and for every $p \in V$,

$$v_p(x) \ge \mu_{Tp}(hx),$$

then T has a linear inverse T^{-1} defined on TV and T^{-1} is strongly bounded.

In normed space, the strongly B-bounded and strongly C-bounded operators with boundedness have the relation as [3] of Theorem 2.6 and 2.7.

Theorem 2.4. (see [3]) Let G be strictly increasing on [0,1]. Then T: $(V, ||.||, G, \alpha) \rightarrow (V', ||.||, G, \alpha)$ is a strongly B-bounded operator if and only if T is a bounded linear operator in a normed space.

Theorem 2.5. (see [3]) Let $T : (V, ||.||, G, \alpha) \to (V', ||.||, G, \alpha)$ be strongly *C*-bounded and let *G* be strictly increasing on [0,1]. Then *T* is a bounded linear operator in a normed space.

Example 2.2. [2] Let (V, ||.||) be a normed space, let G and G' be in $\triangle^+ - \{\epsilon_0, \epsilon_\infty\}$ and consider the identity map I between (V, ||.||, G, M) and (V, ||.||, G', M). Now,

(a) if $G(x_0) = 1$ for some $x \in]0, +\infty[$ while G'(x) < 1 for every $x \in]0, +\infty[$, but $l^-G'(+\infty) = 1$, then I is bounded but not certainly bounded;

(b) if G(x) < 1 for every $x \in]0, +\infty[$, $l^-G(+\infty) = 1$ and $l^-G'(+\infty) < 1$, then I is certainly bounded but not bounded.

From the definition of the bounded linear operator, we can easily know the semi-bounded is not a bounded or a certainly bounded.

Moreover, a linear map T is said to be D-bounded if either (i) or (ii) holds, i.e., if $R_A \in D^+$, then $R_{TA} \in D^+$, where the function R_A defined on R^+ by

$$R_A(x) := \begin{cases} l^- \inf\{v_p(x); p \in A\}, & x \in [0, +\infty[, \\ 1, & x = +\infty. \end{cases}$$
(2.3)

Lemma 2.1. [2] (a) Every strongly bounded operator is also certainly bounded. (b) Every strongly bounded operator is also perhaps bounded. **Theorem 2.6.** (see [2]) Every strongly B-bounded operator is D-bounded.

The identity map I between any PN space (V, v, τ, τ^*) and itself is a strongly bounded operator with k = 1. Also, all linear contraction mappings, according to the definition of [7] are strongly B-bounded. So the identity map I and all linear contraction mappings are also D-bounded.

But the converses need not to be true, i.e., there exists a linear map T is D-bounded but not a strongly B-bounded operator.

Example 2.3. [2] A continuous linear operator is neither certainly bounded nor bounded.

Let (V, ||.||) be a normed space and let F and G be distribution functions in D^+ with $F(x_0) = 1$ for some $x_0 \in]0, +\infty[$. Consider the identity map I from the equilateral space (V, F, M) into the simple space (V, ||.||, G, M). Let A be an unbounded set of (V, ||.||). Then A is certainly bounded in (V, F, M), but A is not D-bounded in (V, ||.||, G, M). Therefore, I is neither certainly bounded nor bounded.

Example 2.4. [2] Let $V = V' = R, v_0 = \mu_0 = \varepsilon_0$. If $p \neq \theta$, then for x > 0 let $v_p(x) = G(\frac{x}{|p|})$ and $\mu_p(x) = U(\frac{x}{|p|})$, where $G(x) = \frac{1}{2}I_{]0,1]}(x) + I_{]1,+\infty]}(x)$.

It is easy to prove that $I: (R, |.|, G, M) \to (R, |.|, U, M)$ is D-bounded but I is not strongly B-bounded, because for every k > 0 and $p \neq 0$, one has, for $x < |p| \min\{\frac{1}{2}, k\}, \ \mu_{Ip}(x) = \mu_p(x) = U(\frac{x}{|p|}) = \frac{x}{|p|} < \frac{1}{2} = G(\frac{x}{|p|}) = v_p(\frac{x}{k}).$

Corollary 2.1. [2] If $T : (V, v, \tau, \tau^*) \to (V', \mu, \sigma, \sigma^*)$ is linear, then T is continuous if and only if it is continuous at θ .

Corollary 2.2. [2] Let $T: (V, v, \tau, \tau^*) \to (V', \mu, \sigma, \sigma^*)$ be a linear onto map with an inverse T^{-1} , if both T and T^{-1} are strongly bounded, then T is a homeomorphism between the PN spaces (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$.

Theorem 2.7. If $T : (V, v, \tau, \tau^*) \to (V', \mu, \sigma, \sigma^*)$ is linear and continuous at θ , then it is uniformly continuous.

Proof. If T is linear, then it is continuous if and only if it is continuous at θ (see Corollary 3.1 of [2]), and by Corollary 3.2 of [2], if T is linear and continuous, then it is uniformly continuous.

Suppose

$$R_A(x) := \begin{cases} l^- \phi_A(x), & x \in [0, +\infty[, \\ 1, & x = +\infty, \end{cases}$$
(2.4)

where $l^-\phi_A(x)$ denotes the left limit of the function f at the point x and $\phi_A(x) = \inf\{v_p(x) : p \in A\}.$

Theorem 2.8. Let (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be two PN spaces and T_1, T_2 be two certainly bounded operators. Then $T_1 + T_2$ is also a certainly bounded operator.

Proof. Since T_1, T_2 are two certainly bounded operators, by its definition, if there exists $x_0 \in (0, +\infty)$ such that $v_p(x_0) = 1$ for all $p \in V$, then there exists $x_1 \in (0, +\infty)$ such that $\mu_{T_1p}(x_1) = 1$ for all $p \in V$; and if there exists $x'_0 \in (0, +\infty)$ such that $v_p(x'_0) = 1$ for all $p \in V$, then there exists $x'_1 \in (0, +\infty)$ such that $\mu_{T_2p}(x'_1) = 1$ for all $p \in V$. Then there exists $x''_1, x''_1 =$ $\max\{x_1, x'_1\}$, so $\mu_{T_1p}(x''_1) = 1$, $\mu_{T_2p}(x''_1) = 1$, for $x \geq x''_1, \mu_{(T_1+T_2)p}(x) =$ $\mu_{T_1p+T_2p}(x) \geq \sigma(\mu_{T_1p}, \mu_{T_2p})(x) \geq \sigma(\mu_{T_1p}, \mu_{T_2p})(x''_1) = 1$, i.e., there exists xsuch that $\mu_{(T_1+T_2)p}(x) = 1$. Hence, $T_1 + T_2$ is certainly bounded operator. \Box

Remark 2.1. If the space is satisfied the condition as the above Theorem 2.4, then $T_1 - T_2$ is also a certainly bounded operator.

Theorem 2.9. Let (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be two PN spaces and $T : V \to V'$ be a certainly bounded operator. Then kT is also a certainly bounded operator for any $k \in R/\{0\}$.

Proof. Since T is a certainly bounded operator, there exists $x_0 \in (0, +\infty)$ such that $v_p(x_0) = 1$ for all $p \in V$. Hence, there exists $x_1 \in (0, +\infty)$ such that $\mu_{Tp}(x_1) = 1$ for all $p \in V$ and for $x > x_1$, $k \in Z/0, \mu_{kTp}(x) \ge \sigma(\mu_{\frac{1}{k}kTp}, \mu_{(1-\frac{1}{k})kTp})(x) = \sigma(\mu_{Tp}, \mu_{(k-1)Tp})(x) = \cdots = \sigma^{k-1}(\mu_{Tp}, \dots, \mu_{Tp})(x) \ge \sigma^{k-1}(\mu_{Tp}, \dots, \mu_{Tp})(x_1) = 1$, i.e., there exists x such that $\mu_{kTp}(x) = 1$. Hence, kT is a certainly bounded operator.

For $|k| \in (0,1]$, $\mu_{kTp}(x) \geq \mu_{Tp}(x)$, and since T is a certainly bounded operator, there exists $x_0 \in (0, +\infty)$ such that $v_p(x_0) = 1$ for all $p \in V$. Hence there exists $x_1 \in (0, +\infty)$ such that $\mu_{Tp}(x_1) = 1$ for $x \geq x_1, \mu_{Tp}(x) \geq \mu_{Tp}(x_1) = 1$. Therefore, $\mu_{kTp}(x) \geq \mu_{Tp}(x) \geq \mu_{Tp}(x_1) = 1$, i.e., $\mu_{kTp}(x) = 1$, kT is a certainly bounded operator.

For |k| > 1, $\mu_{kTp}(x) \ge \mu_{mTp}(x)$, m is an integer number and |m| > |k|, from the above proof, when $m \in Z/0$, we have mT is a certainly bounded operator, so if there exists $x_0 \in (0, +\infty)$ such that $v_p(x_0) = 1$, then there exists $x_1 \in (0, +\infty)$ such that $\mu_{mTp}(x_1) = 1$ for any $x \ge x_1, \mu_{mTp}(x) \ge \mu_{mTp}(x_1) = 1$. Therefore, $\mu_{kTp}(x) \ge \mu_{mTp}(x) \ge \mu_{mTp}(x_1) = 1$ and kT is a certainly bounded operator.

Theorem 2.10. Let (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be two PN spaces, T_1, T_2 be two bounded operators and $T : V \to V'$ be a triangle norm. If the triangle function μ maps $D^+ \times D^+$ into D^+ , i.e., if $\mu(D^+, D^+) \subset D^+$, then $T_1 + T_2$ and kT are also the bounded operators.

Changli Yan

Proof. By Theorem 2.2 of [5], we know that $L_b(V, V')$ is vector subspaces of L(V, V'), where L = L(V, V') is the vector space of linear operators $T : V \to V'$, $L_b = L_b(V, V')$ the subset of L formed by the linear bounded operators from V to V', and $T_1, T_2 \in L_b(V, V')$, so $T_1 + T_2$ and kT are the bounded operators. This completes the proof.

Theorem 2.11. Let (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be two PN spaces, T_1, T_2 be two strongly C-bounded operators and $T : V \to V'$ be a triangle norm. If T = Min, then $T_1 + T_2$ is also a strongly C-bounded operator.

Proof. Since T_1, T_2 are two Strongly C-bounded operators, there exists a constant $h_1 \in (0, 1)$ such that, for every $p \in V$ and for every $x > 0, v_p(x) > 1 - x \Rightarrow \mu_{T_1p}(h_1x) > 1 - h_1x$; and if there exists a constant $h_2 \in (0, 1)$ such that, for every $p \in V$ and for every $x > 0, v_p(x) > 1 - x \Rightarrow \mu_{T_2p}(h_2x) > 1 - h_2x$, then for $h \in (0, 1), \ \mu_{(T_1+T_2)p}(hx) = \mu_{T_1p+T_2p}(hx) \ge \sigma(\mu_{T_1p}, \mu_{T_2p})(hx) = \sigma_T(\mu_{T_1p}, \mu_{T_2p})(hx) = \sup_{\alpha \in [0,1]} T(\mu_{T_1p}(\alpha hx), \mu_{T_2p}((1-\alpha)hx)) \ge T(\mu_{T_1p}(\alpha hx), \mu_{T_2p}((1-\alpha)hx)))$. If $\alpha h = h_1, \ (1-\alpha)h = h_2$, then $T(\mu_{T_1p}(\alpha hx), \mu_{T_2p}((1-\alpha)hx))) \ge T(1 - h_1x, 1 - h_2x) = Min(1 - h_1x, 1 - h_2x) > 1 - hx$. This completes the proof.

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