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# A NONLINEAR ALTERNATIVE IN BANACH ALGEBRAS WITH APPLICATIONS TO FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper a fixed point theorem of Schaefer type involving the product of two operators in a Banach algebra is proved and it is further applied to a first order nonlinear functional differential equation for proving an existence theorem under the mixed generalized Lipschitz and Carathéodory condition.

## 1. Statement of Problem

Let $\mathbf{R}$ denote the real line and let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in $\mathbf{R}$. Let $J = I_0 \cup I$, then $J$ is a closed and bounded interval in $\mathbf{R}$. Let $C$ denote the Banach space of all continuous real-valued functions $\phi$ on $I_0$ with the supremum norm $\|\cdot\|_C$ defined by

$$\|\phi\|_C = \sup_{t \in I_0} |\phi(t)|.$$

Clearly $C$ is a Banach algebra with this norm. Given a function $\phi \in C$, consider the first order functional differential equation (in short FDE)

$$\left.\begin{aligned} \left(\frac{x(t)}{f(t, x_t)}\right)' &= g(t, x_t) \ \text{ a.e. } \ t \in I \\ x(t) &= \phi(t), \ t \in I_0, \end{aligned}\right\} \tag{1.1}$$

where $f : I \times C \to \mathbf{R} - \{0\}$ and $g : I \times C \to \mathbf{R}$ and the function $x_t : I_0 \to \mathbf{R}$ is defined by $x_t(\theta) = x(t + \theta)$ is continuous for each $t \in I$.

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By a solution of FDE (1.1) we mean a function $x \in AC(J, \mathbf{R})$ that satisfies the equations in (1.1), where $AC(J, \mathbf{R})$ is the space of all absolutely continuous real-valued functions on $J$.

The functional differential equations have been the most active area of research since long time. See Hale [11], Henderson [12] and the references therein. But the study of functional differential equations in Banach algebras is very rare in the literature. Very recently the study along this line has been started via fixed point theorems and in Dhage and O'Regan [10] and Dhage [3] some existence results for a particular class of first order functional differential equations have been proved. The FDE (1.1) is new to the literature and the study of this problem will definitely contribute immensely to the area of functional differential equations. The fixed point theorem of Dhage [1] is generally used for proving the existence of solutions under the mixed Lipschitz and Carathéodory conditions. In this article we shall prove the existence theorem for FDE (1.1) using a new nonlinear alternative of Schaefer type to be developed in this paper.

## 2. Auxiliary Results

Let $X$ be a Banach algebra with norm $\|\cdot\|$. A mapping $A : X \to X$is called *$\mathcal{D}$-Lipschitzian* if there exists a continuous and nondecreasing function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ satisfying

$$\|Ax - Ay\| \leq \phi(\|x - y\|) \tag{2.1}$$

for all $x, y \in X$ with $\phi(0) = 0$. Sometimes we call the function $\phi$ to be a *$\mathcal{D}$-function* of the mapping $A$ on $X$. In the special case when $\phi(r) = \alpha r \;\; \alpha > 0$, $A$ is called a Lipschitzian with a Lipschitz constant $\alpha$. In particular if $\alpha < 1, A$ is called a contraction with a contraction constant $\alpha$. Further if $\phi(r) < r$ for $r > 0$, then $A$ is called a nonlinear contraction on $X$.

The following fixed point theorem for a nonlinear contraction is well-known and useful for proving the existence and the uniqueness theorems for nonlinear differential and integral equations.

**Theorem 2.1.** *Let $A : X \to X$ be a nonlinear contraction. Then $A$ has a unique fixed point $x*$ and the sequence $\{A^n x\}$ of successive iterations of $A$ converges to $x*$ for each $x \in X$.*

An operator $T : X \to X$ is called compact if $\overline{T(X)}$ is a compact subset of $X$. Similarly $T : X \to X$ is called totally bounded if $T$ maps a bounded subset of $X$ into the relatively compact subset of $X$. Finally $T : X \to X$ is called completely continuous operator if it is continuous and totally bounded operator on $X$. It is clear that every compact operator is totally bounded , but the converse may not be true. However the two notions are equivalent on a bounded subset of $X$. The details of these concepts may be found in Dhage [7].

The well-known fixed point theorem of Schaefer concerning the completely continuous operators is

**Theorem 2.2.** *Let $T : X \to X$ be a completely continuous operator. Then either*

(i) *the equation $x = \lambda Tx$ has a solution for $\lambda = 1$, or*
(ii) *the set $\mathcal{E} = \{u \in X \mid u = \lambda Tu, 0 < \lambda < 1\}$ is unbounded.*

Theorem 2.2 is extensively used in the theory of nonlinear differential equations for proving the existence results. The method is commonly known as a **priori bound method** for the nonlinear equations. See for example, Dugundji and Granas [9], Zeidler [16] and the references therein. Recently the present author has combined the above two Theorems 2.1 and 2.2 in a Banach algebra and proved the following result.

**Theorem 2.3.** (Dhage [5]) *Let $X$ be a Banach algebra and let $A, B : X \to X$ be two operators satisfying*

(a) *$A$ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\phi$,*
(b) *$B$ is compact and continuous,*
(c) *$M\phi(r) < r$ whenever $r > 0$, where $M = \|B(X)\| = \sup\{\|Bx\| : x \in X\}$.*

*Then either*

(i) *the equation $\lambda A(\frac{x}{\lambda})Bx = x$ has a solution for $\lambda = 1$,or*
(ii) *the set $\mathcal{E} = \{u \in X \mid \lambda A(\frac{u}{\lambda})Bx = u, 0 < \lambda < 1\}$ is unbounded.*

It is known that Theorem 2.3 is useful for proving the existence theorems for the integral equations of mixed type. See Dhage [2] and the references therein. In this paper we shall prove a nonlinear alternative similar to Theorem 2.3 with a slightly different conclusion under the more general conditions via a method different from Dhage [3].

## 3. A Nonlinear Alternative

Before going to the main results we give some preliminaries needed in the sequel. A Kuratowski measure of noncompactness $\alpha$ of a bounded set $A$ in $X$

is a nonnegative real number $\alpha(A)$ defined by

$$\alpha(A) = \inf\{r > 0 : A = \bigcup_{i=1}^{n} A_i,\ \operatorname{diam}(A_i) \leq r,\ \forall i\}. \tag{3.1}$$

The function $\alpha$ enjoys the following properties:

($\alpha_1$) $\alpha(A) = 0 \Longleftrightarrow A$ is relatively compact.

($\alpha_2$) $\alpha(A) = \alpha(\overline{A}) = \alpha(\overline{co}A)$, where $\overline{A}$ and $\overline{co}\,A$ denote respectively the closure and the closed convex hull of $A$.

($\alpha_3$) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

($\alpha_4$) $a(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

($\alpha_5$) $\alpha(\lambda A) = |\lambda|\alpha(A), \forall \lambda \in \mathbf{R}$.

($\alpha_6$) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

The details of measures of noncompactness and their properties appear in Deimling [3] and Zeidler [16].

**Definition 3.1.** A mapping $T : X \to X$ is called *condensing* if for any bounded subset $A$ of $X$, $T(A)$ is bounded and $\alpha(T(A)) < \alpha(A),\ \ \alpha(A) > 0$.

Note that contraction and completely continuous mappings are condensing but the converse may not be true. The following generalization of Theorem 2.2 appears in Martelli [14].

**Theorem 3.1.** *Let $T : X \to X$ be a continuous and condensing operator. Then either*

(i) *the equation $x = \lambda Tx$ has a solution for $\lambda = 1$, or*

(ii) *the set $\mathcal{E} = \{u \in X | u = \lambda Tu, 0 < \lambda < 1\}$ is unbounded.*

Our main result of this section is

**Theorem 3.2.** *Let $X$ be a Banach algebra and let $A, B : X \to X$ be two operators satisfying*

(a) *$A$ is a $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$-function $\phi$,*

(b) *$B$ is compact and continuous,*

(c) *$M\phi(r) < r$ whenever $r > 0$ with $M = \|B(X)\|$.*

*Then either*

(i) *the equation $\lambda AxBx = x$ has a solution for $\lambda = 1$,or*

(ii) *the set $\mathcal{E} = \{u \in X \mid \lambda AuBu = u, 0 < \lambda < 1\}$ is unbounded.*

*Proof.* Define a mapping $T : X \in X$ by

$$Tx = AxBx,\ x \in X. \tag{3.2}$$

Obviously the mapping $T$ is continuous on $X$. The result follows immediately from Theorem 2.2 if the operator $T$ is condensing on $X$. Let $S$ be a and bounded set in $X$. Then we have the following estimate concerning he operators $A$ and $B$. Let $x^*$ be a fixed element of $S$. Then by hypothesis $(a)$,

$$\begin{aligned}\|Ax\| &\leq \|Ax^*\| + \|Ax^* - Ax\| \\ &\leq \|Ax^*\| + \phi(\|x^* - x\|) \\ &\leq \beta\end{aligned}$$

for all $x \in S$, where

$$\beta = \|Ax^*\| + \phi(\text{diam S}) < \infty$$

for all $x \in S$, since $S$ is bounded. Similarly since $B$ is compact, $B(S)$ is a precompact subset of $X$. Hence for $\eta > 0$, there exist subsets $G_1, G_2, \ldots, G_m$ of $X$ such that

$$B(S) = \bigcup_{j=1}^{m} (G_j) \text{ and } \operatorname{diam}(G_j) < \frac{\eta}{\beta}.$$

This further gives that

$$S = \bigcup_{j=1}^{m} B^{-1}(G_j).$$

Let $\epsilon > 0$ be given and suppose that

$$S \subseteq \bigcup_{i=1}^{n} S_i$$

with

$$\operatorname{diam}(S_i) < \alpha(S) + \epsilon$$

for all $i = 1, 2, ..., n$. We put $F_{ij} = S_i \bigcap B^{-1}(G_j)$, then $S \subset \bigcup F_{ij}$.

Now

$$\begin{aligned}T(S) &\subseteq \bigcup_{i,j} T(F_{ij}) \\ &\subset \bigcup_{i,j} T\left(S_i \bigcap B^{-1}(G_j)\right) \\ &= \bigcup_{i,j} Y_{ij}.\end{aligned}$$

If $w_0, w_0 \in Y_{ij}$, for some $i = 1, \ldots, n$ and $j = 1, \ldots, m$, then there exist $x_0, x_1 \in F_{ij} = S_i \bigcap B^{-1}(G_j)$ such that $Tx_0 = w_0$ and $Tx_1 = w_1$.

Since $\phi$ is nondecreasing, one has

$$\begin{aligned}
\|Tx_0 - Tx_1\| &= \|Ax_0Bx_0 - Ax_1 - Ax_2\| \\
&\le \|Ax_0Bx_0 - Ax_1Bx_0\| + \|Ax_1Bx_0 - Ax_1Bx_1\| \\
&\le \|Ax_0 - Ax_1\|\|Bx_0\| + \|Ax_1\|\|Bx_0 - Bx_1\| \\
&\le \phi(\|x_0 - x_1\|)\|Bx_0\| + \|Ax_1\|\|Bx_0 - Bx_1\| \\
&< \phi(\mathrm{diam}(F_{ij}))\|B(X)\| + \|A(S)\|\|Bx_0 - Bx_1\| \\
&\le M\phi(\mathrm{diam}(F_{ij})) + \eta.
\end{aligned}$$

Since $\eta$ is arbitrary, one has

$$\|Tx_0 - Tx_1\| \le M\phi(\mathrm{diam}(F_{ij})).$$

This further implies that

$$\begin{aligned}
\|Tx_0 - Tx_1\| &\le M\phi(\mathrm{diam}(S_i) \\
&< M\phi(\alpha(S) + \epsilon).
\end{aligned}$$

This is true for every $w_0, w_1 \in Y_{ij}$ and so

$$\mathrm{diam}(Y_{ij}) < M\phi(\alpha(S) + \epsilon),$$

for all $i = 1, 2, ..., n$. Thus we have

$$\begin{aligned}
\alpha(T(S)) &= \max_{i,j} \mathrm{diam}(Y_{ij}) \\
&< M\phi(\alpha(S) + \epsilon).
\end{aligned}$$

Since $\epsilon$ is arbitrary, we have

$$\alpha(T(S)) \le M\phi(\alpha(S)) < \alpha(S),$$

whenever $\alpha(S) > 0$.

This shows that $T$ is a condensing on $X$. Now the desired conclusion follows by an application of Theorem 3.1. This completes the proof. $\square$

**Corollary 3.1.** *Let $X$ be a Banach algebra and let $A, B : X \to X$ be two operators satisfying*

(a) *$A$ is a Lipschitzian with a Lipschitz constant $\alpha$,*
(b) *$B$ is compact and continuous,*
(c) *$\alpha M < 1$ where $M = \|B(X)\|$.*

*Then either*

(i) *the equation $\lambda AxBx = x$ has a solution for $\lambda = 1$,or*
(ii) *the set $\mathcal{E} = \{u \in X \mid \lambda AuBu = u, 0 < \lambda < 1\}$ is unbounded.*

In the following section we shall apply our new nonlinear alternative of Dhage-Schaefer type to a nonlinear FDE (1.1) for proving the existence result under some suitable conditions on the functions involved in (1.1).

## 4. Existence Theory

Let $M(J, \mathbf{R})$ and $B(J, \mathbf{R})$ respectively denote the spaces of measurable and bounded real-valued functions on $J$. We shall seek the solution of FDE (1.1) in the space $AC(J, \mathbf{R})$, of all bounded and measurable real-valued functions on $J$. Define a norm $\|\cdot\|_{AC}$ in $AC(J, \mathbf{R})$ by

$$\|x\|_{AC} = \sup_{t \in J} |x(t)|.$$

Clearly $AC(J, \mathbf{R})$ becomes a Banach algebra with this norm. We need the following definition in the sequel.

**Definition 4.1.** A mapping $\beta : I \times C \to \mathbf{R}$ is said to satisfy a condition *of $L^1_X$-Carathéodory* or simply is called *$L^1_X$-Carathéodory* if

(i) $t \mapsto \beta(t, x)$ is measurable for each $x \in C$.
(ii) $x \mapsto \beta(t, x)$ is continuous almost everywhere for $t \in I$,
(iii) there exists a function $h \in L^1(I, \mathbf{R})$ such that

$$|\beta(t, x)| \leq h(t), \quad a.e.\ t \in I$$

for all $x \in C$.

We will need the following hypotheses:

$(H_1)$ The function $f : J \times C \to \mathbf{R}$ is continuous and there exists a function $k \in B(J, \mathbf{R})$ such that $k(t) > 0$, a.e. $t \in J$ and

$$|f(t, x) - f(t, y)| \leq k(t)\|x - y\|_C, \quad a.e.\ t \in I$$

for all $x, y \in C$.

$(H_2)$ $\lim_{t\to 0} f(0, \phi(t)) = 1$.

$(H_3)$ The function $g(t, x)$ is $L^1_X$-Carathéodory.

$(H_4)$ There exists a nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $\gamma \in L^1(I, \mathbf{R})$ such that $\gamma(t) > 0$, a.e. $t \in J$ and

$$|g(t, x)| \leq \gamma(t)\psi\big(\|x\|_C\big), \quad a.e. \ t \in I,$$

for all $x \in C$.

**Theorem 4.1.** *Assume that the hypotheses ($H_1$)-($H_4$) hold. Suppose that*

$$\int_0^\infty \frac{ds}{\psi(s)} > C_1\|\gamma\|_{L^1},$$

*where*

$$C_1 = \frac{\|\phi\| + F}{1 - \|k\|(L + \|h\|_{L^1})}, \|k\|(L + \|h\|_{L^1}) < 1,$$

$F = \max_{t\in J} |f(t, 0)|$, $\|k\| = \max_{t\in J} |k(t)|$ *and* $L = \max\left\{\|\phi\|, \left|\frac{\phi(0)}{f(0,\phi)}\right|\right\}$. *Then the FDE (1.1) has a solution on J.*

*Proof.* Now the FDE (1.1) is equivalent to the functional integral equation (in short FIE)

$$x(t) = [f(t, x_t)]\left(\frac{\phi(0)}{f(0, \phi)} + \int_0^t g(s, x_s)\, ds\right), \ if \ t \in I \tag{4.1}$$

and

$$x(t) = \phi(t), \text{ if } \quad t \in I_0. \tag{4.2}$$

Define the two mappings $A$ and $B$ on $AC(J, \mathbf{R})$ by

$$Ax(t) = \begin{cases} f(t, x_t), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0. \end{cases} \tag{4.3}$$

and

$$Bx(t) = \begin{cases} \frac{\phi(0)}{f(0,\phi)} + \int_0^t g(s, x_s)\, ds, & \text{if } t \in I \\ \phi(t), \quad \text{if } t \in I_0. \end{cases} \tag{4.4}$$

Obviously $A$ and $B$ define the operators $A, B : AC(J, \mathbf{R}) \to AC(J, \mathbf{R})$. Then the FDE (1.1) is equivalent to the operator equation

$$x(t) = Ax(t)Bx(t), \quad t \in J. \tag{4.5}$$

We shall show that the operators $A$ and $B$ satisfy all the hypotheses of Theorem 2.3. We first show that $A$ is a Lipschitzian on $AC(J, \mathbf{R})$.

Let $x, y \in AC(J, \mathbf{R})$. Then by $(H_1)$,

$$\begin{aligned} |Ax(t) - Ay(t)| &\le |f(s, x_t) - f(s, y_t)| \\ &\le k(t)\|x_t - y_t\|_C \\ &\le k(t)\|x - y\|_{AC} \end{aligned}$$

for all $t \in J$. Taking the supremum over $t$ we obtain

$$\|Ax - Ay\|_{AC} \le \|k\|\|x - y\|_{AC}.$$

for all $x, y \in AC(J, \mathbf{R})$. So $A$ is a Lipschitzian on $AC(J, \mathbf{R})$ with a Lipschitz constant $\|k\|$. Next we show that $B$ is completely continuous on $AC(J, \mathbf{R})$. Using the standard arguments as in Granas et al. [10], it is shown that $B$ is a continuous operator on $AC(J, \mathbf{R})$. We shall show that $B(AC(J, \mathbf{R}))$ is a uniformly bounded and equicontinuous set in $AC(J, \mathbf{R})$. Since the function $g$ $L^1_X$-Carathéodory ,we have

$$\begin{aligned} |Bx(t)| &\le L + \int_0^t |g(s, x_s)|\, ds \\ &\le L + \int_0^t h(s)\, ds \\ &\le \text{Ł} + \|h\|_{L^1}, \end{aligned}$$

where $L = \max\left\{\|\phi\|, \left|\frac{\phi(0)}{f(0,\phi)}\right|\right\}$. Taking the supremum over $t$, we obtain $\|Bx\| \le M$ for all $x \in S$, where $M = L + \|h\|_{L^1}$. This shows that $B(AC(J, \mathbf{R}))$ is a uniformly bounded set in $AC(J, \mathbf{R})$. Now we show that $B(AC(J, \mathbf{R}))$ is an equi-continuous set. Let $t, \tau \in I$. Then for any $x \in AC(J, \mathbf{R})$ we have by (4.3),

$$\begin{aligned} |Bx(t) - Bx(\tau)| &\le \left|\int_0^t g(s, x_s)\, ds - \int_0^\tau g(s, x_s)\, ds\right| \\ &\le \left|\int_\tau^t |g(s, x_s)|\, ds\right| \\ &\le \left|\int_\tau^t h(s)\, ds\right| \\ &\le |p(t) - p(\tau)|, \end{aligned}$$

where $p(t) = \displaystyle\int_0^t h(s)\,ds$. Therefore

$$|Bx(t) - Bx(\tau)| \to 0 \quad \text{as} \quad t \to \tau.$$

Again let $\tau \in I_0, t \in I$. Without loss of generality we may assume that $\tau < s$. Then we obtain

$$\begin{aligned}|Bx(t) - Bx(\tau)| &\leq \left|\phi(\tau) - \frac{\phi(0)}{f(0,\phi(s))}\right| + \left|\int_0^t g(s,x_s)\,ds\right|\\ &\leq |\phi(\tau) - \phi(0)| + |p(t) - p(\tau)|\end{aligned}$$

where the function $p$ is defined above. Similarly if $\tau, t \in I_0$, then we get

$$|Bx(t) - Bx(\tau)| \leq |\phi(t) - \phi(\tau)|.$$

Therefore in all above three cases

$$|Bx(t) - Bx(\tau)| \to 0 \text{ as } \quad \tau \to t.$$

Hence $B(AC(J,\mathbf{R}))$ is an equicontinuous set and consequently $B(AC(J,\mathbf{R}))$ is relatively compact by Arzela-Ascoli theorem. Consequently $B$ is a compact and continuous operator on $AC(J,\mathbf{R})$. Thus all the conditions of Theorem 3.1 are satisfied and a direct application of it yields that either conclusion (i) or conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $x \in X$ be any solution to FDE (1.1). Then we have, for any $\lambda \in (0,1)$,

$$\begin{aligned}x(t) &= \lambda Ax(t)Bx(t)\\ &= \begin{cases} \lambda\big[f(t,x_t)\big]\left(\dfrac{\phi(0)}{f(0,\phi)} + \displaystyle\int_0^t g(s,x_s)\,ds\right), \ t \in I\\ \lambda\phi(t), \ t \in I_0\end{cases}\end{aligned}$$

for $t \in J$. Then we have

$$\begin{aligned}|x(t)| &\leq \|\phi\|_C + |f(s,x_t)|\left(L + \left|\int_0^t g(s,x_s)\,ds\right|\right)\\ &\leq \|\phi\|_C + \Big(|f(s,x_t) - f(t,0)| + |f(t,0)|\Big)\left(L + \int_0^t |g(s,x_s)|\,ds\right)\\ &\leq \|\phi\|_C + [k(t)\|x_t\|_C + F]\left(L + \int_0^t |g(s,x_s)|\,ds\right)\\ &\leq \|\phi\|_C + k(t)\|x_t\|_C\left(L + \int_0^t |g(s,x_s)|\,ds\right) + F\int_0^t |g(s,x_s)|\\ &\leq \|\phi\|_C + \|k\|\|x_t\|_C(L + \|h\|_{L^1}) + F\int_0^t \gamma(s)\psi(\|x_s\|_C)\,ds.\end{aligned} \tag{4.6}$$

Put $u(t) = \sup_{s\in[-r,t]} |x(s)|$, for $t \in J$. Then we have

$$|x(t)| \le u(t) \quad \text{and} \quad \|x_t\|_C \le u(t)$$

$\forall t \in J$, and there is a point $t^* \in [-r, t]$ such that $u(t) = |x(t^*)|$.
From (4.6) it follows that

$$\begin{aligned} u(t) &= |x(t^*)| \\ &\le \|\phi\|_C + \|k\| \|x_{t^*}\|_C (L + \|h\|_{L^1}) \\ &\quad + F\left(L + \int_0^{t^*} \gamma(s)\psi(\|x_s\|_C)\, ds\right) \\ &\le \|\phi\|_C + \|k\| u(t)(L + \|h\|_{L^1}) + F\left(L + \int_0^t \gamma(s)\psi(u(s))\, ds\right) \\ &= C_1 + C_2 \int_0^t \gamma(s)\psi(u(s)))\, ds, \end{aligned} \tag{4.7}$$

where

$$C_1 = \frac{\|\phi\|_C + FL}{1 - \|k\|\left[L + \|h\|_L^1\right]} \text{ and } C_2 = \frac{1}{1 - \|k\|\left[L + \|h\|_L^1\right]}.$$

Let

$$w(t) = C_1 + C_2 \int_0^t \gamma(s)\psi(u(s)))\, ds.$$

Then $u(t) \le w(t)$ and a direct differentiation of $w(t)$ yields

$$\left.\begin{aligned} &w'(t) \le C_2\gamma(t)\psi(w(t)) \\ &w(0) = C_1, \end{aligned}\right\} \tag{4.8}$$

that is

$$\begin{aligned} \int_0^t \frac{w'(s)}{\psi(w(s))}\, ds &\le C_2 \int_0^t \gamma(s)]\, ds \\ &\le C_2 \|\gamma\|_{L^1}. \end{aligned}$$

By the change of variables in the above integral gives that

$$\begin{aligned} \int_{C_1}^{w(t)} \frac{ds}{\psi(s)} &\le C_2 \|\gamma\|_{L^1} \\ &< \int_{C_1}^{\infty} \frac{ds}{\psi(s)}. \end{aligned}$$

Now an application of mean value theorem yields that there is a constant $M > 0$ such that $w(t) \leq M$ for all $t \in J$.This further implies that

$$|x(t)| \leq u(t) \leq w(t) \leq M.$$

for all $t \in J$. Thus the conclusion (ii) of Theorem 3.2 does not hold. Therefore the operator equation $AxBx = x$ and consequently the FDE (1.1) has a solution on $J$. This completes the proof. □

## 5. An Example

Let $I_0 = [-\pi/2, 0]$ and $I = [0, \pi/2]$ be two closed and bounded intervals in $\mathbf{R}$, and let $C$ be the space of continuous and $R$-valued functions on $I_0$ with the supremum norm in it. Clearly $C$ is a Banach algebra with respect to the multiplication "$\cdot$" defined by $(x.y)(t) = x(t)y(t)$ for $t \in I_0$. Now consider the nonlinear IVP

$$\left.\begin{array}{c} \left(\dfrac{x(t)}{f(t, x_t)}\right)' = \dfrac{p(t)}{1 + x_t^2}, \ \ a.e. \ \ t \in I \\ x(t) = \sin t, \ \ t \in I_0. \end{array}\right\} \tag{5.1}$$

where $p \in L^1(J, \mathbf{R})$ and $f : J \times C \to \mathbf{R}$ is defined by

$$f(t, x_t) = 1 + \alpha \|x_t\|, \ \ \alpha > 0$$

for all $t \in J$. Obviously $f : J \times C \to \mathbf{R}^+ - \{0\}$. Define a function $g : J \times C \to \mathbf{R}$ by $g(t, x_t) = \dfrac{p(t)}{1 + x_t^2}$. It is easy to verify that $f$ is continuous and Lipschitzian on $J \times C$ with a Lipschitz constant $\alpha$. Further $g(t, x)$ is $L^1_X$-Carathéodory with $h(t) = p(t)$ on $I$. Therefore if $\alpha(1 + \|p\|_{L^1}) < 1$, then by Theorem 4.1, IVP (5.1) has a solution on $J$, because the function $\psi$ satisfies condition (1.1) with $\gamma(t) = p(t), \forall t \in I$ and $\psi(r) = 1 \, \forall \, r \in \mathbf{R}^+$.

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