**Nonlinear Funct. Anal. & Appl., Vol. 9, No. 4 (2004), pp. 649–658**

# AN IMPLICIT ITERATION PROCESS WITH ERRORS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

Chuanzhi Bai and Jong Kyu Kim

Abstract. Let $C$ be a closed convex subset of a real uniformly convex Banach space $E$. Iterative methods for the approximation of common fixed points of a finite family of asymptotically quasi-nonexpansive mappings $T_1, T_2, ..., T_N : C \to C$ are constructed. Our results show that boundedness requirement imposed on the subset $C$ in a result of Sun can be dropped. Furthermore, our results extend the results of Sun to more general iteration methods with errors.

## 1. Introduction

Diaz-Metcalf [3] introduced the concept of quasi-nonexpansive mapping and Goebel-Kirk [5] in 1972 introduced the concept of asymptotically nonexpansive mapping. Let $E$ be a Banach space, $C$ be a nonempty subset of $E$. $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{h_n\}$ in $[0, \infty)$ with $\lim_{n\to+\infty} h_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + h_n)\|x - y\|$$

for all $x, y \in C$ and $n = 1, 2, \cdots$.

$T : C \to C$ is called asymptotically quasi-nonexpansive if there exists a sequence $\{h_n\}$ in $[0, \infty)$ with $\lim_{n\to+\infty} h_n = 0$ such that

$$\|T^n x - q\| \leq (1 + h_n)\|x - q\|$$

Received December 19, 2003.

2000 Mathematics Subject Classification: 46B25, 47H09, 47H10.

Key words and phrases: Implicit iteration process, finite family of asymptotically quasi-nonexpansive mappings, uniform convexity, strong convergence, common fixed point.

The first author was partially supported by the Natural Science Foundation of Jiangsu Education Office (03KJD110056) and the second author was supported by the Kyungnam University Foundation Grant 2004.

for all $x \in C$ and $q \in F(T) = \{x \in C : Tx = x\} \neq \emptyset$ and $n = 1, 2, \cdots$.

$T : C \to C$ is said to be uniformly $L$-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$ and $n = 1, 2, \cdots$.

In [10], Xu-Ori have introduced an implicit iteration process for a finite family of nonexpansive mappings. Recently, Sun [7] extended the process in [10] to a process for a finite family of asymptotically quasi-nonexpansive mappings and proved the following theorem.

**Theorem 1.1 [7].** *Let $E$ be a real uniformly convex Banach space, $C$ be a bounded closed convex subset of $E$. Let $T_i$, $i \in I = \{1, 2, ..., N\}$, be uniformly L-Lipschitzian asymptotically quasi-nonexpansive self-mappings of $C$, i.e., $\|T_i^n x - q_i\| \leq (1 + h_{in})\|x - q_i\|$ for all $x \in C$, $q_i \in F(T_i)$, $i \in I$. Suppose that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, $\sum_{n=1}^{+\infty} h_{in} < +\infty$ for all $i \in I$, and there exists one member $T$ in $\{T_i,\ i \in I\}$ to be semi-compact. Let $x_0 \in C$, and $\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, 1)$. Then the sequence $\{x_n\}$ defined by the following implicit iteration process*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_i^k x_n, \quad n \geq 1, \tag{1.1}$$

*where $n = (k - 1)N + i$, $i \in I$ converges strongly to a common fixed point of the mappings $\{T_i,\ i \in I\}$.*

Theorem 1.1 is a generalization and extension of the corresponding main results in Wittmann [9], Xu-Ori [10].

From Theorem 1.1, two questions arise quite naturally.

**Question 1.** Can the boundedness condition on $C$ in Theorem 1.1 be dropped ?

**Question 2.** Can the implicit iteration process (1.1) in Theorem 1.1 be extended to more general form ?

Inspired and motivated by the recent works in [2, 11], our purpose here is to extend the process (1.1) to a process with errors for a finite family of asymptotically quasi-nonexpansive mappings, with an initial $x_0 \in C$, which is defined as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + \beta_1 T_1 y_1 + \gamma_1 u_1, \\
y_1 &= a_1 x_1 + b_1 T_1 x_1 + c_1 v_1, \\
&\ \ \vdots \\
x_N &= \alpha_N x_{N-1} + \beta_N T_N y_N + \gamma_N u_N,
\end{aligned}$$

$$
\begin{aligned}
&y_N = a_N x_N + b_N T_N x_N + c_N v_N,\\
&x_{N+1} = \alpha_{N+1} x_N + \beta_{N+1} T_1^2 y_{N+1} + \gamma_{N+1} u_{N+1},\\
&y_{N+1} = a_{N+1} x_{N+1} + b_{N+1} T_1^2 x_{N+1} + c_{N+1} v_{N+1},\\
&\qquad\qquad \vdots\\
&x_{2N} = \alpha_{2N} x_{2N-1} + \beta_{2N} T_N^2 y_{2N} + \gamma_{2N} u_{2N},\\
&y_{2N} = a_{2N} x_{2N} + b_{2N} T_N^2 x_{2N} + c_{2N} v_{2N},\\
&\qquad\qquad \vdots
\end{aligned}
$$

which can be written in the following compact form

$$
\begin{aligned}
x_n &= \alpha_n x_{n-1} + \beta_n T_i^k y_n + \gamma_n u_n, \qquad n \geq 1,\\
y_n &= a_n x_n + b_n T_i^k x_n + c_n v_n, \qquad\quad n \geq 1,
\end{aligned}
\tag{1.2}
$$

where $n = (k-1)N + i, i \in I$, $\{u_n\}, \{v_n\}$ are bounded sequence in $C$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = 1$ for $n = 1, 2, \cdots$.

It is our purpose in this paper to give affirmative answers to above two questions. That is, we prove that Theorem 1.1 remains true if process (1.1) be replaced by a process (1.2) and without the boundedness condition imposed on $C$. So, the result presented in this paper is a generalization and extension of the corresponding main results in [6, 7].

## 2. Preliminaries

For convenience, we recall some definitions and conclusions.

**Definition 2.1 [1].** *Let $C$ be a closed subset of a Banach space. A mapping $T : C \to C$ is said be semi-compact, if for any sequence $\{x_n\}$ in $C$ such that $\|x_n - Tx_n\| \to 0$ $(n \to \infty)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^* \in C$.*

Let $E$ be a Banach space. The modulus of convexity of $E$ is the function $\delta_E : [0,2] \to [0,1]$ defined by

$$
\delta_E(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\right\}.
$$

By [4], $\delta_E$ is nondecreasing and

$$
\begin{aligned}
&\|\lambda x + (1-\lambda) y\| \\
&\qquad \leq \max\{\|x\|, \|y\|\}\left[1 - 2\lambda(1-\lambda)\delta_E\left(\frac{\|x - y\|}{\max\{\|x\|, \|y\|\}}\right)\right]
\end{aligned}
\tag{2.1}
$$

for every $x, y \in E \setminus \{0\}$ and $\lambda \in [0,1]$. A Banach space $E$ is called uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

The following lemma is essentially due to Tan-Xu [8, Lemma 1].

**Lemma 2.1 [8].** *Let $\{\rho_n\}$, $\{\lambda_n\}$ and $\{\delta_n\}$ be nonnegative sequences such that for some positive integer $n_0$.*

$$\rho_{n+1} \le (1+\lambda_n)\rho_n + \mu_n, \qquad (n \ge n_0),$$

*where*

$$\sum_{n=n_0}^{+\infty} \lambda_n < +\infty, \qquad \sum_{n=n_0}^{+\infty} \mu_n < +\infty.$$

*Then* $\lim_{n\to+\infty} \rho_n$ *exists.*

## 3. Main results

Now we state and prove the following theorems.

**Theorem 3.1.** *Let $E$ be a real uniformly convex Banach space, $C$ be a closed convex subset of $E$. Let $T_i, i \in I = \{1,2,...,N\}$, be uniformly L-Lipschitzian and asymptotically quasi-nonexpansive self-mappings of $C$ such that $\sum_{n=1}^{+\infty} h_{in} < +\infty$ for all $i \in I$. Let $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of $T_i$, $i \in I$, and there exists one member $T_j$ in $\{T_i, i \in I\}$ to be semi-compact. Let $x_0 \in C$, $\{\beta_n\} \subset [s, 1-s]$ for some $s \in (0,1/2)$, $\limsup_{n\to+\infty} Lb_n < 1$, $\sum_{n=1}^{+\infty} \gamma_n < +\infty$, $\sum_{n=1}^{+\infty} c_n < +\infty$, and $\{u_n\}$, $\{v_n\}$ be arbitrary bounded sequences in $C$. Then the sequence $\{x_n\}$ defined by the implicit iterative process with errors* (1.2) *strongly converges to a common fixed point of the mappings $\{T_i, i \in I\}$.*

*Proof.* For $q \in F$, let $M = \max\{\sup_{n\ge1} \|u_n - q\|, \ \sup_{n\ge1}\|v_n - q\|\}$. It is obvious that $0 < M < +\infty$. Since $T_i$ $(i \in I)$ is asymptotically quasi-nonexpansive, it follows that

$$\begin{aligned}\|y_n - q\| &\le a_n\|x_n - q\| + b_n\|T_i^k x_n - q\| + c_n\|v_n - q\| \\ &\le (1-b_n)\|x_n - q\| + b_n(1+h_{ik})\|x_n - q\| + c_n M \\ &= (1+b_n h_{ik})\|x_n - q\| + c_n M,\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\|x_n - q\| &\le \alpha_n\|x_{n-1} - q\| + \beta_n\|T_i^k y_n - q\| + \gamma_n\|u_n - q\| \\ &\le \alpha_n\|x_{n-1} - q\| + (1-\alpha_n)(1+h_{ik})\|y_n - q\| + \gamma_n M.\end{aligned} \tag{3.2}$$

Substitute (3.1) into (3.2) to get

$$
\begin{aligned}
\|x_n - q\| &\le \alpha_n \|x_{n-1} - q\| + (1-\alpha_n)(1+h_{ik})(1+b_n h_{ik})\|x_n - q\| \\
&\quad + ((1-\alpha_n)(1+h_{ik})c_n + \gamma_n)M \\
&\le \alpha_n \|x_{n-1} - q\| + (1-\alpha_n + h_{ik})(1+b_n h_{ik})\|x_n - q\| \\
&\quad + ((1+h_{ik})c_n + \gamma_n)M \qquad (3.3)\\
&\le \alpha_n \|x_{n-1} - q\| + (1-\alpha_n + (2+h_{ik})h_{ik})\|x_n - q\| \\
&\quad + ((1+h_{ik})c_n + \gamma_n)M.
\end{aligned}
$$

Since $\lim_{n\to\infty} \gamma_n = 0$, there exists a natural number $n_1$ such that for $n > n_1$, $\gamma_n \le s/2$. So $\alpha_n = 1 - \beta_n - \gamma_n \ge 1 - (1-s) - s/2 = s/2$ for $n > n_1$. Thus, we have by (3.3) that

$$
\begin{aligned}
\|x_n - q\| \le \|x_{n-1} - q\| + \frac{2(2+h_{ik})h_{ik}}{s}\|x_n - q\| \qquad (3.4)\\
+ \frac{2((1+h_{ik})c_n + \gamma_n)M}{s}, \quad (n > n_1).
\end{aligned}
$$

Since $\sum_{k=1}^{+\infty} h_{ik} < +\infty$ for all $i \in I$, $\lim_{k\to\infty} h_{ik} = 0$ for each $i \in I$. Hence, there exists a natural number $n_2$ with $k > n_2/N + 1$ such that for $n > n_2$, $h_{ik} \le \sqrt{1+s/6} - 1$, $i \in I$. Obviously, $\forall i \in I$, $(2+h_{ik})h_{ik} \le s/6$ for $n > n_2$. Let $n_3 = \max\{n_1, n_2\}$. Then, for (3.4) becomes

$$
\begin{aligned}
&\|x_n - q\| \\
&\le \frac{s}{s - 2(2+h_{ik})h_{ik}}\|x_{n-1} - q\| + 2\frac{(1+h_{ik})c_n + \gamma_n}{s - 2(2+h_{ik})h_{ik}}M \qquad (3.5)\\
&= \left(1 + \frac{2(2+h_{ik})}{s - 2(2+h_{ik})h_{ik}}h_{ik}\right)\|x_{n-1} - q\| + 2\frac{(1+h_{ik})c_n + \gamma_n}{s - 2(2+h_{ik})h_{ik}}M \\
&\le \left(1 + \frac{3(1+\sqrt{1+s/6})}{s}h_{ik}\right)\|x_{n-1} - q\| + 3\frac{\sqrt{1+s/6}\,c_n + \gamma_n}{s}M \\
&:= (1+\lambda_{n-1})\|x_{n-1} - q\| + \mu_{n-1}, \quad (n > n_3).
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
\sum_{n=n_3+1}^{+\infty} \lambda_{n-1} &= \frac{3(1+\sqrt{1+s/6})}{s}\sum_{n=n_3+1}^{+\infty} h_{ik} \\
&\le \frac{3(1+\sqrt{1+s/6})}{s}\sum_{i=1}^{N}\sum_{n=[(n_3+1)/N]}^{+\infty} h_{ik} < +\infty,
\end{aligned}
$$

and

$$\sum_{n=n_3+1}^{+\infty} \mu_{n-1} = \frac{3}{s} M \left( \sqrt{1+s/6} \sum_{n=n_3+1}^{+\infty} c_n + \sum_{n=n_3+1}^{+\infty} \gamma_n \right) < +\infty.$$

From Lemma 2.1 we have $\lim_{n\to\infty} \|x_n - q\|$ exists. If $\lim_{n\to\infty} \|x_n - q\| = 0$, then the conclusion of Theorem 3.1 holds. So, we assume that $\lim_{n\to\infty} \|x_n - q\| := d > 0$. Then, there exists a natural number $n_0$ such that $n > n_0$, $2d \geq \|x_n - q\| \geq d/2$. Let $n_4 = \max\{n_0, n_3\}$. Combining (3.1) and (3.5), then we have for all $n > n_4$,

$$\begin{aligned}
&\|T_i^k y_n - q\| \\
&\leq (1+h_{ik})\|y_n - q\| \leq (1+h_{ik})^2 \|x_n - q\| + (1+h_{ik}) c_n M \\
&\leq (1+h_{ik})^2 \left( 1 + \frac{3(1+\sqrt{1+s/6})}{s} h_{ik} \right) \|x_{n-1} - q\| \\
&\quad + 3(1+h_{ik})^2 M \frac{\sqrt{1+s/6}\, c_n + \gamma_n}{s} + (1+h_{ik}) c_n M \\
&= \left[ 1 + \left( 2 + h_{ik} + (1+h_{ik})^2 \frac{3(1+\sqrt{1+s/6})}{s} \right) h_{ik} \right] \|x_{n-1} - q\| \\
&\quad + 3(1+h_{ik})^2 M \frac{\sqrt{1+s/6}\, c_n + \gamma_n}{s} + (1+h_{ik}) c_n M \qquad (3.6) \\
&\leq \left[ 1 + \left( 1 + \sqrt{1+s/6} + \left(1 + \frac{s}{6}\right) \frac{3(1+\sqrt{1+s/6})}{s} \right) h_{ik} \right] \|x_{n-1} - q\| \\
&\quad + 3\left(1 + \frac{s}{6}\right) M \frac{\sqrt{1+s/6}\, c_n + \gamma_n}{s} + \sqrt{1+s/6}\, c_n M \\
&:= (1 + m_1(s) h_{ik}) \|x_{n-1} - q\| + m_2(s) c_n + \left( \frac{1}{2} + \frac{3}{s} \right) M \gamma_n,
\end{aligned}$$

where

$$m_1(s) = \frac{3(s+2)}{2s} \left( 1 + \sqrt{1+s/6} \right)$$

and

$$m_2(s) = \frac{3(s+2)}{2s} \sqrt{1+s/6} M.$$

By (2.1) and (3.6), we obtain for each $n > n_4 + 1$,

$$
\begin{aligned}
&\|x_n - q\| \\
&= \|(1-\beta_n)(x_{n-1} - q) + \beta_n(T_i^k y_n - q) - \gamma_n(x_{n-1} - q) + \gamma_n(u_n - q)\| \\
&\le \|(1-\beta_n)(x_{n-1} - q) + \beta_n(T_i^k y_n - q)\| + \gamma_n(\|x_{n-1} - q\| + \|u_n - q\|) \\
&\le \max\{\|x_{n-1} - q\|,\ \|T_i^k y_n - q\|\} \\
&\quad \times \left[1 - 2\beta_n(1-\beta_n)\delta_E\left(\frac{\|T_i^k y_n - x_{n-1}\|}{\max\{\|x_{n-1} - q\|,\ \|T_i^k y_n - q\|\}}\right)\right] \\
&\quad + (2d + M)\gamma_n \qquad\qquad (3.7) \\
&\le \left((1 + m_1(s)h_{ik})\|x_{n-1} - q\| + m_2(s)c_n + \left(\frac{1}{2} + \frac{3}{s}\right) M\gamma_n\right) \\
&\quad \times \left[1 - 2s^2\delta_E\left(\frac{\|T_i^k y_n - x_{n-1}\|}{\max\{\|x_{n-1} - q\|,\ \|T_i^k y_n - q\|\}}\right)\right] \\
&\quad + (2d + M)\gamma_n.
\end{aligned}
$$

Since $\delta_E$ is nondecreasing and $h_{ik} \le \sqrt{1 + s/6} - 1 \le \sqrt{s/6}$, using (3.7), we have for all $n > n_4 + 1$,

$$
\begin{aligned}
&s^2 d\,\delta_E\left(\frac{\|T_i^k y_n - x_{n-1}\|}{2(1 + m_1(s)\sqrt{s/6})d + m_2(s) + (1/2 + 3/s)M}\right) \\
&\le \left((1 + m_1(s)h_{ik})\|x_{n-1} - q\| + m_2(s)c_n + \left(\frac{1}{2} + \frac{3}{s}\right) M\gamma_n\right) \\
&\quad \times 2s^2\delta_E\left(\frac{\|T_i^k y_n - x_{n-1}\|}{\max\{\|x_{n-1} - q\|,\ \|T_i^k y_n - q\|\}}\right) \\
&\le (1 + m_1(s)h_{ik})\|x_{n-1} - q\| - \|x_n - q\| + m_2(s)c_n \\
&\quad + \left(2d + \frac{3(s+2)}{2s} M\right)\gamma_n \qquad\qquad (3.8) \\
&= \|x_{n-1} - q\| - \|x_n - q\| + m_1(s)h_{ik}\|x_{n-1} - q\| \\
&\quad + m_2(s)c_n + \left(2d + \frac{3(s+2)}{2s} M\right)\gamma_n \\
&\le \|x_{n-1} - q\| - \|x_n - q\| + 2dm_1(s)h_{ik} \\
&\quad + m_2(s)c_n + \left(2d + \frac{3(s+2)}{2s} M\right)\gamma_n.
\end{aligned}
$$

By (3.8), we conclude that for all $m > n_4 + 2$,

$$
\begin{aligned}
s^2 d \sum_{n=n_4+2}^{m} \delta_E\left(\frac{\|T_i^k y_n - x_{n-1}\|}{2(1+m_1(s)\sqrt{s/6})d + m_2(s) + (1/2+3/s)M}\right) \\
\le \|x_{n_4+1} - q\| + 2dm_1(s)\sum_{n=n_4+2}^{m} h_{ik} + m_2(s)\sum_{n=n_4+2}^{m} c_n \\
+ \left(2d + \frac{3(s+2)}{2s}M\right)\sum_{n=n_4+2}^{m} \gamma_n.
\end{aligned}
$$

Hence

$$
\sum_{n=n_4+2}^{\infty} \delta_E\left(\frac{\|T_i^k y_n - x_{n-1}\|}{2(1+m_1(s)\sqrt{s/6})d + m_2(s) + (1/2+3/s)M}\right) < \infty
$$

and thus

$$
\lim_{n\to\infty} \delta_E\left(\frac{\|T_i^k y_n - x_{n-1}\|}{2(1+m_1(s)\sqrt{s/6})d + m_2(s) + (1/2+3/s)M}\right) = 0.
$$

Since $E$ is a real uniformly convex Banach space, it follows that

$$
\lim_{n\to+\infty} \|T_i^k y_n - x_{n-1}\| = 0.
$$

Hence

$$
\begin{aligned}
\|x_n - x_{n-1}\| &\le \beta_n \|T_n^k y_n - x_{n-1}\| + \gamma_n \|u_n - x_{n-1}\| \\
&\le \beta_n \|T_n^k y_n - x_{n-1}\| + \gamma_n (\|u_n - q\| + \|x_{n-1} - q\|) \\
&\le \beta_n \|T_n^k y_n - x_{n-1}\| + \gamma_n (M + 2d) \to 0 \quad (n\to\infty),
\end{aligned}
$$

as well as

$$
\|x_n - x_{n+l}\| \to 0 \quad (n\to\infty),
$$

for all $l < 2N$. For convenience, let $\sigma_n = \|T_i^k x_n - x_{n-1}\|$. Since $T_n = T_i$ for $n = (k-1)N + i$, we have $\sigma_n = \|T_n^k x_n - x_{n-1}\|$. Noticing that

$$
\begin{aligned}
\|T_n^k x_n - x_{n-1}\| &\le \|T_n^k x_n - T_n^k y_n\| + \|T_n^k y_n - x_{n-1}\| \\
&\le L\|x_n - y_n\| + \|T_n^k y_n - x_{n-1}\| \\
&\le L(b_n \|T_n^k x_n - x_n\| + c_n \|v_n - x_n\|) + \|T_n^k y_n - x_{n-1}\| \\
&\le Lb_n (\|T_n^k x_n - x_{n-1}\| + \|x_n - x_{n-1}\|) + \|T_n^k y_n - x_{n-1}\| \\
&\quad + Lc_n (\|v_n - q\| + \|x_n - q\|) \\
&\le Lb_n \|T_n^k x_n - x_{n-1}\| + Lb_n \|x_n - x_{n-1}\| + \|T_n^k y_n - x_{n-1}\| \\
&\quad + Lc_n (M + 2d), \quad (n > n_4).
\end{aligned}
$$

Thus by $\limsup_{n\to\infty} Lb_n < 1$, we get that

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \|T_n^k x_n - x_{n-1}\| = 0.$$

Therefore, as in [7], for $n > N$,

$$\|x_{n-1} - T_n x_n\| \le \sigma_n + L^2\|x_n - x_{n-N}\| + L\sigma_{n-N} + L\|x_n - x_{(n-N)-1}\|,$$

which yields that $\lim_{n\to+\infty} \|x_{n-1} - T_n x_n\| = 0$. From

$$\|x_n - T_n x_n\| \le \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|,$$

it follows that

$$\lim_{n\to+\infty} \|x_n - T_n x_n\| = 0.$$

Furthermore, as in [7], we obtain

$$\lim_{n\to+\infty} \|x_n - T_l x_n\| = 0, \quad (l \in I). \tag{3.9}$$

By hypothesis, we may assume that $T_1$ is semi-compact without loss of generality. Therefore by (3.9) it follows that $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$ and by the definition of semi-compactness, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in C$. By (3.9) again, we obtain that $x^* \in F$. Replace $q$ by $x^*$ in (3.5), from Lemma 2.1 , we easily known that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. So $\lim_{n\to\infty} \|x_n - x^*\| = \lim_{j\to\infty} \|x_{n_j} - x^*\| = 0$, that is $\{x_n\}$ converges to a common fixed point $x^*$ in $F$. This completes the proof. □

**Remark.** If we set $\gamma_n = 0$, $b_n = c_n = 0$ in Theorem 3.1, then we obtain the main result of [7] (Theorem 3.3) without the boundedness condition imposed on the subset $C$.

**Theorem 3.2.** *Let $E$ be a real uniformly convex Banach space, $C$ be a closed convex subset of $E$. Let $T_i, i \in I$, be asymptotically nonexpansive self-mappings of $C$ such that $\sum_{n=1}^{+\infty} h_{in} < +\infty$ for all $i \in I$. Let $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of $T_i$, $i \in I$, and there exists one member $T_j$ in $\{T_i, i \in I\}$ to be semi-compact. Let $x_0 \in C$, $\{\beta_n\} \subset [s, 1-s]$ for some $s \in (0, 1/2)$, $\limsup_{n\to+\infty} Lb_n < 1$ ($L = \sup_{i\in I, n\ge 1}\{1 + h_{in}\}$), $\sum_{n=1}^{+\infty} \gamma_n < +\infty$, $\sum_{n=1}^{+\infty} c_n < +\infty$, and $\{u_n\}$, $\{v_n\}$ be arbitrary bounded sequences in $C$. Then the sequence $\{x_n\}$ defined by the implicit iterative process with errors* (1.2) *strongly converges to a common fixed point of the mappings $\{T_i, i \in I\}$.*

## References

1. S. S. Chang, Y. J. Cho, J. K. Kim, and K. H. Kim, *Iterative approximation of fixed points for asymptotically nonexpansive type mappings in Banach spaces*, PanAmer. Math. Jour. **11** (2001), 53-63.
2. C. E. Chidume, *Convergence theorems for asymptotically pseudocontractive mappings*, Nonlinear Analysis TMA **49** (2002), 1-11.
3. J. B. Diaz and F. T. Metcalf, *On the structure of the set of subsequential limit points of successive approximations*, Bull. Amer. Math. Soc. **73** (1967), 516-519.
4. J. Diestel, *Geometry of Banach Space-selected topics, Lecture Notes in Math.***485**, Springer-Verlag, Berlin, Heidelberg, New York, 1975.
5. K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171-174.
6. J. K. Kim, K. H. Kim and K. S. Kim, *Convergence theorems of modified three-step iterative sequences with mixed errors for asymptotically quasi-nonexpansive mappings in Banach spaces*, PanAmer. Math. Jour. **14** (2004), 45-54.
7. Z. H. Sun, *Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings*, J. Math. Anal. Appl. **286** (2003), 351-358.
8. K. K. Tan and H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), 301-308.
9. R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58** (1992), 486-491.
10. H. K. Xu and R. G. Ori, *An implicit iteration process for nonexpansive mappings*, Numer. Funct. Anal. Optim. **22** (2001), 767-773.
11. Y. Xu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive equations*, J. Math. Anal. Appl. **224** (1998), 91-101.

Chuanzhi Bai
Department of Mathematics
Huaiyin Normal College
Huaian, Jiangsu 223001
P. R. China
*E-mail address*: czbai8@sohu.com

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan, Kyungnam 631-701
Korea
*E-mail address*: jongkyuk@kyungnam.ac.kr