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INEQUALITIES FOR SOLUTIONS TO SOME NONLINEAR EQUATIONS

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Abstract. Let F be a nonlinear Fréchet differentiable map in a real Hilbert space. Condition sufficient for existence of a solution to the equation $F(u) = 0$ is given, and a method (dynamical systems method, DSM) to calculate the solution as the limit of the solution to a Cauchy problem is justified under suitable assumptions.

1. Introduction

In this paper a method is given for proving existence of a solution to a nonlinear operator equation $F(u) = 0$ in a Hilbert space and for computing this solution. Our method (the dynamical systems method: DSM) consists of solving a suitable Cauchy problem which has a global solution $u(t)$ such that $y := u(\infty)$ does exist and $F(y) = 0$. In [8], global convergence of a Newton-type DSM method is proved for solvable operator equations with C^2-monotone operators in a Hilbert space, and estimates for the solution are obtained. The results of the present paper generalize some results in [1] and [2]. Examples of applications of DSM and its development for nonlinear ill-posed problems, for problems with unbounded operators, and for construction of convergent iterative schemes for nonlinear ill-posed operator equations are given in [1],[7],[8]-[15]. In [2]-[4] continuous regularization methods are used for a study of well-posed operator equations with smooth operators and for some equations with monotone operators. In [7]-[10] and [15] DSM was developed for solving ill-posed operator equations, with not necessarily monotone operators, and for constructing convergent iterative methods for their solution.

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Let F be a nonlinear Fréchet differentiable map in a real Hilbert space. Consider the equation

$$F(u) = 0. \tag{1.1}$$

Let $\Phi(t, u)$ be a map, continuous with respect to t in the norm of H and Lipschitz with respect to u in the ball $B := \{u : \|u - u_0\| \leq R,\ u \in H\}$. Weaker conditions, which guarantee local existence and uniqueness of the solution to (1.6) below, would suffice. Assume that:

$$(F'(u)\Phi(t, u), F(u)) \leq -g_1(t)\|F(u)\|^2 \quad \forall u \in B, \tag{1.2}$$

and

$$\|\Phi(t, u)\| \leq g_2(t)\|F(u)\| \quad \forall u \in B, \tag{1.3}$$

where $g_j, j = 1, 2$, are positive functions on $R_+ := [0, \infty)$, g_2 is continuous, $g_1 \in L^1_{loc}(R_+)$,

$$\int_0^\infty g_1 dt = +\infty, \tag{1.4}$$

and

$$G(t) := g_2(t) \exp(-\int_0^t g_1 ds) \in L^1(R_+). \tag{1.5}$$

Remark. *Sometimes the assumption* (1.3) *can be used in the following more general form*:

$$\|\Phi(t, u)\| \leq g_2(t)\|F(u)\|^b \quad \forall u \in B, \tag{1.3'}$$

where $b > 0$ is a constant. The statements and proofs of Theorems 1-3 *in Sections* 1 *and* 2 *can be easily adjusted to this assumption.*

Consider the following Cauchy problem:

$$\dot{u} = \Phi(t, u), \quad u(0) = u_0, \quad \dot{u} := \frac{du}{dt}. \tag{1.6}$$

Assume that

$$\|F(u_0)\| \int_0^\infty G(t)dt \leq R. \tag{1.7}$$

The above assumptions (1.2)-(1.5) and (1.7) on F, Φ, g_1 and g_2 hold throughout and are not repeated in the statement of Theorem 1 below. By global solution to (1.6) we mean a solution defined for all $t \geq 0$.

Theorem 1. *Under the above assumptions problem* (1.6) *has a global solution* $u(t)$, *there exists the strong limit* $y := \lim_{t\to\infty} u(t) = u(\infty)$, $F(y) = 0$, $u(t) \in B$ *for all* $t \geq 0$, *and the following inequalities hold*:

$$\|u(t) - y\| \leq \|F(u_0)\| \int_t^\infty G(x)dx, \tag{1.8}$$

and

$$\|F(u(t))\| \leq \|F(u_0)\| \exp(-\int_0^t g_1(x)dx). \tag{1.9}$$

In Section 2 proof of Theorem 1 is given, and two other theorems are proved, and in Section 3 examples of applications are presented. In Section 4 a linear equation and in Section 5 a nonlinear equation are discussed.

2. Proof of Theorem 1 and additional results

The assumptions about Φ imply local existence and uniqueness of the solution $u(t)$ to (1.6). To prove global existence of u, it is sufficient to prove a uniform with respect to t bound on $\|u(t)\|$. Indeed, if the maximal interval of the existence of $u(t)$ is finite, say $[0, T)$, and $\Phi(t, u)$ is locally Lipschitz with respect to u, then $\|u(t)\| \to \infty$ as $t \to T$.

Let $g(t) := \|F(u(t))\|$. Since H is real, one uses (1.6) and (1.2) to get $g\dot{g} = (F'(u)\dot{u}, F) \leq -g_1(t)g^2$, so $\dot{g} \leq -g_1(t)g$, and integrating one gets (1.9), because $g(0) = \|F(u_0)\|$. Using (1.3), (1.6) and (1.9), one gets:

$$\|u(t) - u(s)\| \leq g(0) \int_s^t G(x)dx, \quad G(x) := g_2(x) \exp(-\int_0^x g_1(z)dz). \tag{2.1}$$

Because $G \in L^1(R_+)$, it follows from (2.1) that the limit $y := \lim_{t\to\infty} u(t) = u(\infty)$ exists, and $y \in B$ by (1.7). Inequality (1.9) and the continuity of F imply $F(y) = 0$, so y solves (1.1). Taking $t \to \infty$ and setting $s = t$ in (2.1) yields estimate (1.8). The inclusion $u(t) \in B$ for all $t \geq 0$ follows from (2.1) and (1.7). Theorem 1 is proved. □

If condition (1.2) is replaced by

$$(F'\Phi, F) \leq -g_1(t)\|F\|^a, \quad 0 < a < 2, \tag{2.2}$$

then the proof of Theorem 1 yields the inequality $g^{1-a}\dot{g} \leq -g_1(t)$. So

$$0 \leq g(t) \leq [g^{2-a}(0) - (2-a) \int_0^t g_1(s)ds]^{\frac{1}{2-a}}. \tag{2.3}$$

If (1.4) holds, then (2.3) implies $g(t) = 0$ for all $t \geq T$, where T is defined by the equation:

$$g^{2-a}(0) - (2-a)\int_0^T g_1(s)ds = 0, \quad 0 < a < 2. \tag{2.4}$$

Thus $\|F(u(t))\| = 0$ for $t \geq T$. So, by (1.3), $\Phi = 0$ for $t \geq T$. Thus, by (1.6), $u(t) = u(T)$ for $t \geq T$. Therefore $y := u(T)$ solves equation (1.1), $F(y) = 0$, and $\|u(T) - u(0)\| \leq \|F(u_0)\| \int_0^T g_2 ds$. If $\|F(u_0)\| \int_0^T g_2 ds \leq R$, then $u(t) \in B$ for all $t \geq 0$. We have proved:

Theorem 2. *If* (1.2) *is replaced by* (2.2), (1.4) *holds, and* $\|F(u_0)\| \int_0^T g_2 ds \leq R$, *where* T *is defined by* (2.4), *then equation* (1.1) *has a solution in* $B = \{u : \|u - u_0\| \leq R\}$, *the solution* $u(t)$ *to* (1.6) *exists for all* $t > 0$, $u(t) \in B$, $u(t) := y$ *for* $t \geq T$, *and* $F(y) = 0$, $y \in B$.

If (2.2) holds with $a > 2$, and (1.4) holds, then a similar calculation yields:

$$0 \leq g(t) \leq [g^{-(a-2)}(0) + (a-2)\int_0^t g_1(s)ds]^{\frac{1}{2-a}} := h(t) \to 0 \quad t \to \infty, \tag{2.5}$$

because of (1.4). Assume that

$$\int_0^\infty g_2(s)h(s)ds \leq R. \tag{2.6}$$

Then (1.3) and (1.6) yield

$$\|u(t) - u(0)\| \leq R,$$

and

$$\|u(t) - u(\infty)\| \leq \int_t^\infty g_2(s)h(s)ds \to 0 \text{ as } t \to \infty.$$

Therefore an analog of Theorem 1 is obtained:

Theorem 3. *If* (2.2) *holds with* $a > 2$, *and assumptions* (1.4) *and* (2.6) *hold, then the solution* $u(t)$ *to (1.6) exists for all* $t > 0$, $u(t) \in B$, *there exists* $u(\infty) := y$, *and* $F(y) = 0$, *where* $y \in B$.

3. Applications

If $g_j = c_j$, $j = 1, 2$, and $c_j > 0$ are constants, then (1.4) and (1.5) hold, $\int_0^\infty G dx = c_2/c_1$, so condition (1.7) is:

$$\frac{c_2}{c_1}\|F(u_0)\| \le R, \quad c_j = c_j(R),\ j = 1, 2. \tag{3.1}$$

Let us give examples of applications of Theorem 1 using its simplified version with $g_j = c_j > 0$, $j = 1, 2$.

Example 1. Continuous Newton-type method.
Let $\Phi = -[F'(u)]^{-1}F(u)$, and assume

$$\|[F'(u)]^{-1}\| \le m_1 = m_1(R), \quad \forall u \in B. \tag{3.2}$$

Assumption (3.2) holds in all the examples below. It implies "well-posedness" of equation (1.1). Under the above assumptions one has $c_1 = 1$, $c_2 = m_1$. The operator Φ is locally Lipschitz if one assumes

$$\|F''(u)\| \le M_2, \quad \forall u \in B, \tag{3.3}$$

where $M_2 = M_2(R)$ is a positive constant. Condition (3.1) takes the form:

$$m_1(R)\|F(u_0)\| \le R. \tag{3.4}$$

In the examples below condition (3.3) is assumed and not repeated.

Conclusion 1. *By Theorem* 1, *inequality* (3.4) *implies existence of a solution* y *to* (1.1) *in* B, *global existence and uniqueness of the solution* $u(t)$ *to* (1.6), *convergence of* $u(t)$ *to* y *as* $t \to \infty$, *and the error estimate* (1.9). *Condition* (3.4) *is always satisfied if equation* (1.1) *has a solution* y *and if* u_0 *is chosen sufficiently close to* y.

Example 2. Continuous simple iterations method.
Let $\Phi = -F$, and assume $F'(u) \ge c_1(R) > 0$ for all $u \in B$. Then $c_2 = 1$, $c_1 = c_1(R)$, and (3.1) is:

$$[c_1(R)]^{-1}\|F(u_0)\| \le R. \tag{3.5}$$

If this inequality holds, then Conclusion 1 *holds with* (3.5) *replacing* (3.4).

Example 3. Continuous gradient method.

Let $\Phi = -[F']^*F$ and assume (3.2). Then $c_2 = M_1(R)$, because $\|[F'(u)]^*\| = \|F'(u)\| \leq M_1(R)$, and $(F'\Phi, F) = -\|[F'(u)]^*F\|^2 \leq -m_1^{-2}\|F\|^2$, so $c_1 = m_1^{-2}$, where m_1 is the constant from (3.2). Here we have used the estimates $\|f\| = \|A^{-1}Af\| \leq \|A^{-1}\|\|Af\|, \|Af\| \geq \|A^{-1}\|^{-1}\|f\|$, with $A := F'(u)$ and $f = F(u)$. Estimate (3.1) is:

$$M_1 m_1^2 \|F(u_0)\| \leq R. \tag{3.6}$$

If this inequality holds, then Conclusion 1 *holds with* (3.6) *replacing* (3.4).

Example 4. Continuous Gauss-Newton method.

Let $\Phi = -([F']^*F')^{-1}[F']^*F$, and assume (3.2). Then $c_1 = 1$, $c_2 = m_1^2 M_1$, and (3.1) is:

$$M_1 m_1^2 \|F(u_0)\| \leq R. \tag{3.7}$$

If this inequality holds, then Conclusion 1 *holds with* (3.7) *replacing* (3.4).

Example 5. Continuous modified Newton method.

$\Phi = -[F'(u_0)]^{-1}F(u)$, and assume $\|[F(u_0)]^{-1}\| \leq m_0$. Then $c_2 = m_0$. To find c_1, let us note that:

$$(F'\Phi, F) = -\|F(u)\|^2 - ((F'(u) - F'(u_0))[F(u_0)]^{-1}F, F) \leq 0.5\|F(u)\|^2,$$

provided that

$$|((F'(u) - F'(u_0))[F(u_0)]^{-1}F, F)| \leq M_2 R m_0 \|F(u)\|^2 \leq 0.5\|F(u)\|^2,$$

that is, $M_2 R m_0 \leq 0.5$. If $R = (2M_2 m_0)^{-1}$, then the last inequality becomes an equality. Choosing such R, one has $c_2 = m_0$, $c_1 = 0.5$, and (3.1) is: $2m_0\|F(u_0)\| \leq (2M_2 m_0)^{-1}$, that is,

$$4m_0^2 M_2 \|F(u_0)\| \leq 1. \tag{3.8}$$

If this inequality holds, then Conclusion 1 *holds with* (3.8) *replacing* (3.4).

Example 6. Descent methods.

Let $\Phi = -\frac{f}{(f',h)}h$, where $f = f(u(t))$ is a differentiable functional $f : H \to [0, \infty)$, and h is an element of H. One has $\dot{f} = (f', \dot{u}) = -f$. Thus $f = f_0 e^{-t}$, where $f_0 := f(u_0)$. Assume $\|\Phi\| \leq c_2|f|^b$, $b > 0$. Then $\|\dot{u}\| \leq c_2|f_0|^b e^{-bt}$. Therefore $u(\infty)$ does exist, $f(u(\infty)) = 0$, and $\|u(\infty) - u(t)\| \leq ce^{-bt}$, $c = const > 0$.

If $h = f'$, and $f = \|F(u)\|^2$, then $f'(u) = 2[F']^*(u)F(u)$, $\Phi = -\frac{f}{\|f'\|^2}f'$, and (1.6) is a descent method. For this Φ one has $c_1 = \frac{1}{2}$, and $c_2 = \frac{m_1}{2}$, where m_1 is defined in (3.2). Condition (3.1) in this example is condition (3.4).

If (3.4) *holds, then Conclusion* 1 *holds.*

In Example 6 some results from [4] are obtained. Our approach is more general than the one in [4], since the choices of f and h do not allow one, e.g., to obtain Φ used in Example 5.

4. Remark about linear equations

The following result was proved in [2]: if equation $Ay = f$ in a Hilbert space has a solution y, and $A \geq 0$ is a linear selfadjoint operator, then the global solution u to the regularized Cauchy problem

$$\dot{u} = -Au - \alpha(t)u + f, \quad u(0) = u_0, \tag{4.1}$$

has a limit $\lim_{t\to\infty} u(t) := u(\infty)$, and $A(u(\infty)) = f$. In [2] $u_0 \in H$ is arbitrary, $\alpha > 0$ is a continuously differentiable, monotonically decaying to zero as $t \to \infty$, function on R_+, $\int_0^\infty \alpha dt = +\infty$, and $\alpha^{-2}\dot{\alpha} \to 0$ as $t \to \infty$.

If $\alpha > 0$, $\dot{\alpha} \leq 0$, and $\alpha^{-2}|\dot{\alpha}| \leq c$, where $c = const$, then $\alpha^{-1}(t) - \alpha^{-1}(0) \leq ct$, so $\alpha(t) \geq [ct + \alpha^{-1}(0)]^{-1}$ and consequently $\int_0^\infty \alpha dt = +\infty$. Therefore the condition $\int_0^\infty \alpha dt = +\infty$ in [2] can be dropped.

In this Section we give a new derivation of the result in [2] under weaker assumptions about α, and show that the regularization in (4.1) is not necessary.

First, let us prove that the regularization in (4.1) is not necessary: the result holds with $\alpha = 0$.

Below $\to$ denotes strong convergence in H.

The solution to (4.1) with $\alpha = 0$ is

$$u(t) = U(t)u_0 + \int_0^t U(t-s)f ds,$$

where $U(t) := \exp(-tA)$. If E_λ is the resolution of the identity of the selfadjoint operator A, then $U(t)u_0 = \int_0^\infty e^{-t\lambda} dE_\lambda u_0 \to Pu_0$ as $t \to \infty$, where P is the operator of the orthogonal projection on N, and N is the null-space of A. Also $\int_0^t U(t-s)f ds = \int_0^\infty (1 - e^{-t\lambda}) dE_\lambda y \to y - Py$ as $t \to \infty$, by the dominated convergence theorem. Thus, $u(\infty) = y - Py + Pu_0$ and $A(u(\infty)) = f$. $\square$

Consider now the case $0<\alpha\to 0$ as $t\to\infty$. If $h(t):=\exp(\int_0^t\alpha(s)ds)$, and u solves (4.1), then

$$u(t)=h^{-1}(t)U(t)u_0+h^{-1}(t)\int_0^\infty \exp(-t\lambda)\int_0^t e^{s\lambda}h(s)ds\lambda dE_\lambda y. \qquad (4.2)$$

Using L'Hospital's rule one gets

$$\lim_{t\to\infty}\frac{\lambda\int_0^t e^{s\lambda}h(s)ds}{e^{t\lambda}h(t)}=\lim_{t\to\infty}\frac{\lambda e^{t\lambda}h(t)}{\lambda e^{t\lambda}h(t)+e^{t\lambda}h(t)\alpha(t)}=1 \quad \forall\lambda>0. \qquad (4.3)$$

From (4.2), (4.3), and the dominated convergence theorem, one gets $u(\infty)=y-Py$. The first term on the right-hand side of (4.2) tends to zero as $t\to\infty$ (even if $Pu_0\neq 0$), if $h(\infty)=\infty$. To apply the dominated convergence theorem, one checks that

$$\frac{\lambda\int_0^t e^{-(t-s)\lambda}h(s)ds}{h(t)}=\frac{\lambda\int_0^t e^{-s\lambda}h(t-s)ds}{h(t)}\leq 1$$

for all $t>0$ and all $\lambda>0$, where the inequality $0<h(t-s)\leq h(t)$, valid for $s\geq 0$, was used. □

Our derivation uses less restrictive assumptions on α than in [2]: we do not assume differentiability of α, and the property $\lim_{t\to\infty}\alpha^{-2}\dot\alpha=0$. The property $\int_0^\infty\alpha dt=+\infty$, which is equivalent to $h(\infty)=\infty$, was used above only to prove that $\lim_{t\to\infty}h^{-1}(t)U(t)u_0=0$. If $\int_0^\infty\alpha dt:=q<\infty$, then $u(\infty)=y-Py+e^{-q}Pu_0$, and $Au(\infty)=f$, so that the basic conclusions hold without the assumption $h(\infty)=\infty$.

Finally, let us prove a *typical for ill-posed problems result: the rate of convergence $u(t)\to y$ can be as slow as one wishes, it is not uniform with respect to f.*

Let us assume $\alpha=0$, but the proof is essentially the same for $0<\alpha\to 0$ as $t\to\infty$. Assume that $A>0$ is compact, and $A\varphi_j=\lambda_j\varphi_j$, $(\varphi_j,\varphi_m)=\delta_{jm}$. Then (4.2) with $y=y_m:=\varphi_m$ and $u_0=0$ yields $u(t)=\varphi_m(1-e^{-t\lambda_m})$. Thus $u(\infty)=y$, but for any fixed $T>0$, however large, one can find m such that $\|u(T)-y_m\|>0.5$, that is, the convergence is not uniform with respect to f.

5. Remark about nonlinear equations

In this Section we give a short and simple proof of the basic result in [3], and close a gap in the proof in [3], where it is not explained why one can apply the L'Hospital's rule the second time. A completely different approach to a study of operator equations with non-smooth monotone operators is given in [13].

The assumptions in [3] are: *the operator* A *is monotone* (*possibly nonlinear*), *defined on all of* H, *hemicontinuous, problem* (4.1) *has a unique global solution, equation* $A(y) = f$ *has a solution,* $\alpha(t) > 0$ *decays monotonically to zero,* $\lim_{t\to\infty} \dot{\alpha}\alpha^{-2} = 0$, *and* α *is convex.*

We refer below to these assumptions as A3). If A3) hold, the basic result, proved in [3], is the existence of $u(\infty) := y$, and the relation $A(y) = f$. In [3], p. 184, under the additional assumption, (namely, assumption (1.24) from [3]), the global existence of the solution to (4.1) is proved. Actually, the assumption about global existence of the solution to (4.1) can be dropped altogether: in [5], p.99, it is proved that A3) (and even weaker assumptions) imply that problem (4.1) has a unique global solution.

Let us give a proof of the basic result from [3]. It is well known that A3) imply that the problem $A(v_\alpha) + \alpha v_\alpha - f = 0$, for any fixed number $\alpha > 0$, has a unique solution, there exists $\lim_{\alpha\to 0} v_\alpha := y$, $A(y) = f$, and $\|y\| \le \|z\|$, for any $z \in \{z : A(z) = f\}$. Thus, for any small $\delta > 0$, one can find α_δ such that $\|v_\alpha - y\| < \delta/2$ for all $\alpha > \alpha_\delta$, $\lim_{\delta\to 0} \alpha_\delta = 0$. Let $w := u - v_\alpha$, where u solves (4.1) and v_α does not depend on t. Then $\dot{w} = -[A(u) - A(v_\alpha) + \alpha(t)(u - v_\alpha) + (\alpha(t) - \alpha)v_\alpha]$. Multiply this by w, use the monotonicity of A, and let $\|w\| := g$. Then $g\dot{g} \le -\alpha(t)g^2 + c|\alpha(t) - \alpha|g$, $c = \|y\|$.

Multiply $A(v_\alpha) + \alpha v_\alpha - A(y) = 0$ by $v_\alpha - y$ and use monotonicity of A to get $\alpha(v_\alpha, v_\alpha - y) \le 0$. Thus $\|v_\alpha\| \le \|y\|$, so $c = \|y\|$.

Since $\alpha(t)$ is convex, one has $|\alpha(t) - \alpha| \le |\dot{\alpha}(t)|(t_\alpha - t)$, where $t_\alpha \ge t$ is defined by the equation $\alpha = \alpha(t_\alpha)$, $\lim_{\alpha\to 0} t_\alpha = \infty$. Thus, $g\dot{g} \le -\alpha(t)g^2 + c|\dot{\alpha}(t)|(t_\alpha - t)g$, and, taking $u(0) = v_\alpha$, one gets

$$g(t_\alpha) \le ce^{-\int_0^{t_\alpha} \alpha(x)dx} \int_0^{t_\alpha} e^{\int_0^s \alpha(x)dx} |\dot{\alpha}(s)|(t_\alpha - s)ds. \tag{5.1}$$

We *claim* (and prove below) that

$$\lim_{t\to\infty} \alpha(t)e^{\int_0^t \alpha(s)ds} = \infty. \tag{5.2}$$

This allows one to apply twice L'Hospital's rule to the right-hand side of (5.1), and to get:

$$\lim_{\alpha\to 0} g(t_\alpha) = \lim_{t_\alpha\to\infty} \frac{\dot{\alpha}(t_\alpha)}{\dot{\alpha}(t_\alpha) + \alpha^2(t_\alpha)} = 0.$$

Now,

$$\|u(t_\alpha) - y\| \le \|u(t_\alpha) - v_\alpha\| + \|v_\alpha - y\|, \quad \|v_\alpha - y\| \le \delta/2,$$

and $\|u(t_\alpha) - v_\alpha\| \le \delta/2$, for sufficiently large t_α. Since $\delta > 0$ is arbitrarily small, it follows that $\lim_{t\to\infty} \|u(t) - y\| = 0$.

Let us prove *claim* (5.2). From our assumptions about α, it follows that for all sufficiently large t, one has $-\dot{\alpha}\alpha^{-2} \le c$, where $0 < c < 1$, so $\alpha(t) \ge (c_1+t)^{-1}b$, where $b := c^{-1} > 1$, $c_1 > 0$ is a constant, and $e^{\int_0^t \alpha(s)ds} \ge (c_1+t)^b$. Thus, (5.2) holds, and the basic result from [3] is obtained. □

If one assumes additionally that A is Fréchet differentiable, then the proof is shorter. Namely, let $h(t) := \|A(u(t)) + \alpha(t)u(t) - f\| := \|\psi\|$. Then $h\dot{h} = -((A'(u(t)) + \alpha(t))\psi, \psi) \le -\alpha(t)h^2$, because $A' \ge 0$ due to the monotonicity of A. Thus $h(t) \le \phi(t)$, where $\phi(t) := h(0)e^{-\int_0^t \alpha ds}$. As we proved in Section 4, the assumptions on $\alpha(t)$ imply $\alpha(t) \ge (c_1 t + c_2)^{-1}$, where c_1 and c_2 are positive constants, and c_1 can be chosen so that $0 < c_1 < 1$, due to the assumption $\lim_{t\to\infty} \dot{\alpha}\alpha^{-2} = 0$. Therefore $\int_0^\infty \phi(t)dt < \infty$. From (4.1) one gets: $\|\dot{u}\| \le \phi(t)$. Because $\int_0^\infty \phi(t)dt < \infty$, it follows that $u(\infty) := y$ exists, and $\|u(\infty) - u(t)\| \le \int_t^\infty \phi(s)ds$. Finally, $A(y) = f$ because $h(\infty) = 0 = \|A(y) - f\|$. Any choice of α, for which $\int_0^\infty \phi(t)dt < \infty$, is sufficient for the above argument. □

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