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FUZZY WAVELET TYPE OPERATORS

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Abstract. The basic wavelet type operators A_k, B_k, C_k, D_k, $k \in \mathbb{Z}$ were studied extensively in the real case (see [2]). Here they are extended to the fuzzy setting and are defined similarly via a real valued scaling function. Their pointwise and uniform convergence with rates to the fuzzy unit operator I is established. The produced Jackson type inequalities involve the fuzzy first modulus of continuity and usually are proved to be sharp, in fact attained. Furthermore all fuzzy wavelet type operators A_k, B_k, C_k, D_k preserve monotonicity in the fuzzy sense. Here we do not assume any kind of orthogonality condition on the scaling function φ, and the operators act on fuzzy valued continuous functions over $\mathbb{R}$.

1. Background

Definition 1.1 ([8]). Let $\mu: \mathbb{R} \to [0,1]$ with the following properties:

(i) μ is *normal*, i.e., $\exists x_0 \in \mathbb{R}: \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$, $\forall x, y \in \mathbb{R}$, $\forall \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is *upper semicontinuous* on $\mathbb{R}$, i.e., $\forall x_0 \in \mathbb{R}$ and $\forall \varepsilon > 0$, $\exists$ neighborhood $V(x_0): \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall x \in V(x_0)$.

(iv) The set $\overline{\operatorname{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\operatorname{supp}(\mu) := \{x \in \mathbb{R};\ \mu(x) > 0\}$).

We call μ a *fuzzy real number*. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\mathcal{X}_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\mathcal{X}_{\{x_0\}}$ is the characteristic function at x_0.

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For $0 < r \le 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R}: \mu(x) \ge r\}$ and

$$[\mu]^0 := \overline{\{x \in \mathbb{R}: \mu(x) > 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see [8]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \le r_1 \le r_2 \le 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = [u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} \le u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall r \in [0,1]$.

Define

$$D: \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max\{|u_-^{(r)} - v_-^{(r)}|, |u_+^{(r)} - v_+^{(r)}|\},$$

where $[v]^r = [v_-^{(r)}, v_+^{(r)}]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [8], with the properties

$$\begin{aligned} D(u \oplus w, v \oplus w) &= D(u,v), \quad \forall u, v, w \in \mathbb{R}_{\mathcal{F}}, \\ D(k \odot u, k \odot v) &= |k| D(u,v), \quad \forall u, v \in \mathbb{R}_{\mathcal{F}}, \ \forall k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) &\le D(u,w) + D(v,e), \ \forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}. \end{aligned}$$

Let $f, g: \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be *fuzzy real number values functions*. The distance between f, g is defined by

$$D^*(f,g) := \sup_{x \in \mathbb{R}} D(f(x), g(x)).$$

On $\mathbb{R}_{\mathcal{F}}$ we define a *partial order by* "$\le$": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \le v$ iff $u_-^{(r)} \le v_-^{(r)}$ and $u_+^{(r)} \le v_+^{(r)}$, $\forall r \in [0,1]$.

Lemma 1.2. ([4]). *For any $a, b \in \mathbb{R}: a, b \ge 0$ and any $u \in \mathbb{R}_{\mathcal{F}}$ we have*

$$D(a \odot u, b \odot u) \le |a - b| \cdot D(u, \tilde{o}),$$

where $\tilde{o} \in \mathbb{R}_{\mathcal{F}}$ is defined by $\tilde{o} := \mathcal{X}_{\{0\}}$.

Lemma 1.3. ([4]).

(i) *If we denote* $\tilde{o} := \mathcal{X}_{\{0\}}$, *then* $\tilde{o} \in \mathbb{R}_{\mathcal{F}}$ *is the neutral element with respect to* $\oplus$, *i.e.,* $u \oplus \tilde{o} = \tilde{o} \oplus u = u$, $\forall u \in \mathbb{R}_{\mathcal{F}}$.

(ii) *With respect to* $\tilde{o}$, *none of* $u \in \mathbb{R}_{\mathcal{F}}$, $u \neq \tilde{o}$ *has opposite in* $\mathbb{R}_{\mathcal{F}}$.

(iii) *Let* $a, b \in \mathbb{R}$: $a \cdot b \geq 0$, *and any* $u \in \mathbb{R}_{\mathcal{F}}$. *Then we have* $(a+b) \odot u = a \odot u \oplus b \odot u$. *For general* $a, b \in \mathbb{R}$, *the above property is fail.*

(iv) *For any* $\lambda \in \mathbb{R}$ *and any* $u, v \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda \oplus (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.

(v) *For any* $\lambda, \mu \in \mathbb{R}$ *and* $u \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda \odot (\mu \odot u) = (\lambda \cdot \mu) \odot u$.

(vi) *If we denote* $\|u\|_{\mathcal{F}} := D(u, \tilde{o})$, $\forall u \in \mathbb{R}_{\mathcal{F}}$, *then* $\|\cdot\|_{\mathcal{F}}$ *has the properties of a usual norm on* $\mathbb{R}_{\mathcal{F}}$, *i.e.,*

$$\|u\|_{\mathcal{F}} = 0 \textit{ iff } u = \tilde{o}, \|\lambda \odot u\|_{\mathcal{F}} = |\lambda| \cdot \|u\|_{\mathcal{F}},$$
$$\|u \oplus v\|_{\mathcal{F}} \leq \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}, \ \|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}} \leq D(u, v).$$

Notice that $(\mathbb{R}_{\mathcal{F}}, \oplus, \odot)$ is *not* a linear space over $\mathbb{R}$, and consequently $(\mathbb{R}_{\mathcal{F}}, \|\cdot\|_{\mathcal{F}})$ is *not* a normed space.

Here $\sum^*$ denotes the fuzzy summation.

We need also a particular case of the *Fuzzy Henstock integral* $\left(\delta(x) = \frac{\delta}{2}\right)$ introduced in [8], Definition 2.1.

That is,

Definition 1.4. ([6]). Let $f: [a, b] \to \mathbb{R}_{\mathcal{F}}$. We say that f is *Fuzzy–Riemann integrable* to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u, v]; \xi\}$ of $[a, b]$ with the norms $\Delta(P) < \delta$, we have

$$D\left(\sum_P{}^* (v - u) \odot f(\xi), I\right) < \varepsilon.$$

We choose to write

$$I := (FR) \int_a^b f(x) dx.$$

We also call an f as above (FR)-*integrable.*

Theorem 1.5. ([7]). *Let* $f: [a, b] \to \mathbb{R}_{\mathcal{F}}$ *be fuzzy continuous. Then* $(FR)\int_a^b f(x)dx$ *exists and belongs to* $\mathbb{R}_{\mathcal{F}}$, *furthermore it holds*

$$\left[(FR) \int_a^b f(x) dx\right]^r = \left[\int_a^b (f)_-^{(r)}(x) dx, \int_a^b (f)_+^{(r)}(x) dx\right], \quad \forall r \in [0, 1].$$

Denote by $C(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ the space of fuzzy continuous functions and by $C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ the space of bounded fuzzy continuous functions on $\mathbb{R}$ with respect to metric D.

Lemma 1.6. ([3]). *If $f, g\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ are fuzzy continuous functions, then the function $F\colon [a,b] \to \mathbb{R}_+$ defined by $F(x) := D(f(x), g(x))$ is continuous on $[a,b]$, and*

$$D\left((FR)\int_a^b f(u)du, (FR)\int_a^b g(u)du\right) \leq \int_a^b D(f(x), g(x))dx.$$

Definition 1.7. ([3]). Let $f\colon \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy real number valued function. We define the (*first*) *fuzzy modulus of continuity of* f by

$$\omega_1^{(\mathcal{F})}(f, \delta) := \sup_{\substack{x,y\in\mathbb{R}\\ |x-y|\leq\delta}} D(f(x), f(y)), \quad \delta > 0.$$

Definition 1.8. ([3]). Let $f\colon \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. We call f a *uniformly continuous fuzzy real number valued function*, iff for any $\varepsilon > 0$ there exists $\delta > 0$: whenever $|x - y| \leq \delta$; $x, y \in \mathbb{R}$, implies that $D(f(x), f(y)) \leq \varepsilon$. We denote it as $f \in C_{\mathcal{F}}^U(\mathbb{R})$.

Proposition 1.9. ([3]). *Let $f \in C_{\mathcal{F}}^U(\mathbb{R})$. Then $\omega_1^{(\mathcal{F})}(f, \delta) < +\infty$, any $\delta > 0$.*

Proposition 1.10. ([3]). *It holds*

(i) $\omega_1^{(\mathcal{F})}(f, \delta)$ *is nonnegative and nondecreasing in* $\delta > 0$, *any* $f\colon \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$.
(ii) $\lim_{\delta\downarrow 0} \omega_1^{(\mathcal{F})}(f, \delta) = \omega_1^{(\mathcal{F})}(f, 0) = 0$, *iff* $f \in C_{\mathcal{F}}^U(\mathbb{R})$.
(iii) $\omega_1^{(\mathcal{F})}(f, \delta_1 + \delta_2) \leq \omega_1^{(\mathcal{F})}(f, \delta_1) + \omega_1^{(\mathcal{F})}(f, \delta_2)$, $\delta_1, \delta_2 > 0$, *any* $f\colon \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$.
(iv) $\omega_1^{(\mathcal{F})}(f, n\delta) \leq n\omega_1^{(\mathcal{F})}(f, \delta)$, $\delta > 0$, $n \in \mathbb{N}$, *any* $f\colon \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$.
(v) $\omega_1^{(\mathcal{F})}(f, \lambda\delta) \leq \lceil\lambda\rceil\omega_1^{(\mathcal{F})}(f, \delta) \leq (\lambda + 1)\omega_1^{(\mathcal{F})}(f, \delta)$, $\lambda > 0$, $\delta > 0$, *where* $\lceil\cdot\rceil$ *is the ceiling of the number, any* $f\colon \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$.
(vi) $\omega_1^{(\mathcal{F})}(f \oplus g, \delta) \leq \omega_1^{(\mathcal{F})}(f, \delta) + \omega_1^{(\mathcal{F})}(g, \delta)$, $\delta > 0$, *any* $f, g\colon \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$.
(vii) $\omega_1^{(\mathcal{F})}(f, \cdot)$ *is continuous on* $\mathbb{R}_+$, *for* $f \in C_{\mathcal{F}}^U(\mathbb{R})$.

2. Results

Now, we present our first main result.

Theorem 2.1. *Let $f \in C(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ and the scaling function $\varphi(x)$ a real valued bounded function with supp $\varphi(x) \subseteq [-a, a]$, $0 < a < +\infty$, $\varphi(x) \geq 0$, such that $\sum_{j=-\infty}^{\infty} \varphi(x - j) \equiv 1$ on $\mathbb{R}$. For $k \in \mathbb{Z}$, $x \in \mathbb{R}$ put*

$$(B_k f)(x) := \sum_{j=-\infty}^{\infty}{}^{*} f\left(\frac{j}{2^k}\right) \odot \varphi(2^k x - j), \tag{1}$$

which is a fuzzy wavelet type operator. Then

$$D(B_k f)(x), f(x)) \leq \omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^k}\right), \tag{2}$$

and

$$D^*(B_k f, f) \leq \omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^k}\right), \tag{3}$$

all $x \in \mathbb{R}$, and $k \in \mathbb{Z}$. If $f \in C_{\mathcal{F}}^U(\mathbb{R})$, then as $k \to +\infty$ we get $\omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^k}\right) \to 0$ and $\lim\limits_{k\to+\infty} B_k f = f$, pointwise and uniformly with rates.

Proof. Notice that

$$(B_k f)(x) = \sum_{\substack{j \\ 2^k x - j \in [a,a]}}^{*} f\left(\frac{j}{2^k}\right) \odot \varphi(2^k x - j).$$

We would like to estimate

$$\begin{aligned}
&D((B_k f)(x), f(x)) \\
&= D\left(\sum_{\substack{j \\ 2^k x - j \in [-a,a]}}^{*} f\left(\frac{j}{2^k}\right) \odot \varphi(2^k x - j), f(x) \odot 1\right) \\
&= D\left(\sum_{\substack{j \\ 2^k x - j \in [-a,a]}}^{*} f\left(\frac{j}{2^k}\right) \odot \varphi(2^k x - j), f(x) \odot \sum_{j=-\infty}^{\infty} \varphi(2^k x - j)\right) \\
&= D\left(\sum_{\substack{j \\ 2^k x - j \in [-a,a]}}^{*} f\left(\frac{j}{2^k}\right) \odot \varphi(2^k x - j), \quad \sum_{\substack{j \\ 2^k x - j \in [-a,a]}}^{*} f(x) \odot \varphi(2^k x - j)\right) \\
&\leq \sum_{\substack{j \\ 2^k x - j \in [-a,a]}} \varphi(2^k x - j) D\left(f\left(\frac{j}{2^k}\right), f(x)\right) \\
&\leq \sum_{\substack{j \\ 2^k x - j \in [-a,a]}} \varphi(2^k x - j) \omega_1^{(\mathcal{F})}\left(f, \left|\frac{j}{2^k} - x\right|\right)
\end{aligned}$$

$$\left(\text{here } x - \frac{j}{2^k} \in \left[-\frac{a}{2^k}, \frac{a}{2^k}\right]\right)$$

$$\leq \left(\sum_{\substack{j \\ 2^k x - j \in [-a,a]}} \varphi(2^k x - j) \right) \omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^k}\right) = 1 \cdot \omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^k}\right).$$

□

It follows the next important result.

Theorem 2.2. *Let $f \in C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ and the scaling function $\varphi(x)$ a real valued function with supp $\varphi(x) \subseteq [-a, a]$, $0 < a < +\infty$, φ is continuous on $[-a, a]$, $\varphi(x) \geq 0$, such that $\sum_{j=-\infty}^{\infty} \varphi(x - j) = 1$ on $\mathbb{R}$ (then $\int_{-\infty}^{\infty} \varphi(x)dx = 1$). Define*

$$\varphi_{kj}(t) := 2^{k/2}\varphi(2^k t - j), \quad \textit{for } k, j \in \mathbb{Z}, \quad t \in \mathbb{R}, \tag{4}$$

$$\langle f, \varphi_{kj} \rangle := (FR) \int_{\frac{j-a}{2^k}}^{\frac{j+a}{2^k}} f(t) \odot \varphi_{kj}(t)dt, \tag{5}$$

and set

$$(A_k f)(x) := \sum_{j=-\infty}^{\infty}{}^{*} \langle f, \varphi_{kj} \rangle \odot \varphi_{kj}(x), \quad x \in \mathbb{R}, \tag{6}$$

which a fuzzy wavelet type operator. Then

$$D((A_k f)(x), f(x)) \leq \omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^{k-1}}\right), \quad x \in \mathbb{R}, \ k \in \mathbb{Z}, \tag{7}$$

and

$$D^*((A_k f), f) \leq \omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^{k-1}}\right). \tag{8}$$

If $f \in C_{\mathcal{F}}^U(\mathbb{R})$ and bounded, then again we get $A_k \to$ unit operator I with rates as $k \to +\infty$.

Proof. Since φ is compactly supported we have

$$\varphi_{kj}(t) \neq 0 \text{ iff } -a \leq 2^k t - j \leq a, \text{ iff } \frac{j-a}{2^k} \leq t \leq \frac{j+a}{2^k}.$$

Also it holds that

$$(A_k f)(x) := \sum_{\substack{j \\ 2^k x - j \in [-a,a]}}^{*} \langle f, \varphi_{kj} \rangle \odot \varphi_{kj}(x), \quad k \in \mathbb{Z}.$$

We would like to estimate

$$D((A_k f)(x), f(x)) = D\left(\sum^{*}_{\substack{j \\ 2^k x - j \in [-a,a]}} \langle f, \varphi_{kj} \rangle \odot \varphi_{kj}(x), f(x) \right)$$

$$= D\left(\sum^{*}_{\substack{j \\ 2^k x - j \in [-a,a]}} \langle f, \varphi_{kj} \rangle \odot \varphi_{kj}(x), f(x) \odot \sum_{\substack{j \\ 2^k x - j \in [-a,a]}} \varphi(2^k x - j) \right)$$

$$= D\left(\sum^{*}_{\substack{j \\ 2^k x - j \in [-a,a]}} \langle f, \varphi_{kj} \rangle \odot \varphi_{kj}(x), \sum^{*}_{\substack{j \\ 2^k x - j \in [-a,a]}} f(x) \odot 2^{-k/2} \varphi_{kj}(x) \right)$$

$$\leq \sum_{\substack{j \\ 2^k x - j \in [-a,a]}} \varphi_{kj}(x) D(\langle f, \varphi_{kj} \rangle, 2^{-k/2} \odot f(x)) =: K_1.$$

Next we estimate separately

$$D(\langle f, \varphi_{kj} \rangle, 2^{-k/2} \odot f(x))$$

$$= D\left((FR) \int_{\frac{j-a}{2^k}}^{\frac{j+a}{2^k}} f(t) \odot \varphi_{kj}(t) dt, 2^{-k/2} \odot f(x) \right)$$

$$= D\left(2^{k/2} \odot (FR) \int_{\frac{j-a}{2^k}}^{\frac{j+a}{2^k}} f(t) \odot \varphi(2^k t - j) dt, 2^{-k/2} \odot f(x) \right)$$

(in Fuzzy-Riemann integral we can have linear change of variables)

$$= D\left(2^{k/2} \odot (FR) \int_{j-a}^{j+a} f\left(\frac{u}{2^k} \right) \odot \varphi(u-j) \frac{du}{2^k}, 2^{-k/2} \odot f(x) \right)$$

$$= D\left(2^{-k/2} \odot (FR) \int_{j-a}^{j+a} f\left(\frac{u}{2^k} \right) \odot \varphi(u-j) du, 2^{-k/2} \odot f(x) \right)$$

$$= 2^{-k/2} D\left((FR) \int_{j-a}^{j+a} f\left(\frac{u}{2^k} \right) \odot \varphi(u-j) du, f(x) \odot 1 \right) =: K_2.$$

Notice that $\int_{-\infty}^{\infty} \varphi(u-j) du = 1$, $j \in \mathbb{Z}$ and by compact support of φ we have

$$\int_{j-a}^{j+a} \varphi(u-j) du = 1.$$

Hence

$$K_2 = 2^{-k/2} D\left((FR)\int_{j-a}^{j+a} f\left(\frac{u}{2^k}\right) \odot \varphi(u-j)du, f(x) \odot \int_{j-a}^{j+a} \varphi(u-j)du\right)$$

$$= 2^{-k/2} D\left((FR)\int_{j-a}^{j+a} f\left(\frac{u}{2^k}\right) \odot \varphi(u-j)du, (FR)\int_{j-a}^{j+a} f(x) \odot \varphi(u-j)du\right)$$

(by Lemma 1.6) and

$$\le 2^{-k/2} \int_{j-a}^{j+a} D\left(f\left(\frac{u}{2^k}\right) \odot \varphi(u-j), f(x) \odot \varphi(u-j)\right) du$$

(by Lemma 2.3 next)

$$= 2^{-k/2} \int_{j-a}^{j+a} \varphi(u-j) D\left(f\left(\frac{u}{2^k}\right), f(x)\right) du$$

$$\le 2^{-k/2} \int_{j-a}^{j+a} \varphi(u-j) \omega_1^{(\mathcal{F})}\left(f, \left|\frac{u}{2^k} - x\right|\right) du$$

$\left(\text{notice that } -\dfrac{a}{2^{k-1}} \le \dfrac{u}{2^k} - x \le \dfrac{a}{2^{k-1}}\right)$

$$\le 2^{-k/2} \left(\int_{j-a}^{j+a} \varphi(u-j)du\right) \omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^{k-1}}\right) \le 2^{-k/2} \omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^{k-1}}\right).$$

That is, we prove that

$$D(\langle f, \varphi_{kj}\rangle, 2^{-k/2} \odot f(x)) \le 2^{-k/2} \omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^{k-1}}\right).$$

Hence, we get

$$K_1 \le \sum_{\substack{j \\ 2^k x - j \in [-a,a]}} \varphi_{kj}(x) 2^{-k/2} \omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^{k-1}}\right)$$

$$= \left(\sum_{\substack{j \\ 2^k x - j \in [-a,a]}} \varphi(2^k x - j)\right) \omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^{k-1}}\right)$$

$$= \left(\sum_{j=-\infty}^{\infty} \varphi(2^k x - j)\right) \omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^{k-1}}\right) = 1 \cdot \omega_1^{(\mathcal{F})}\left(f, \frac{a}{2^{k-1}}\right), \; x \in \mathbb{R}. \quad \square$$

Here we use the following lemma.

Lemma 2.3. *Let $f: \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ fuzzy continuous and bounded, i.e. $\exists M_1 > 0: D(f(x), \tilde{o}) \le M_1$, $\forall x \in \mathbb{R}$. Let also $g: J \subseteq \mathbb{R} \to \mathbb{R}_+$ continuous and bounded, i.e. $\exists M_2 > 0: g(x) \le M_2$, $\forall x \in J$, where J is an interval. Then $f(x) \odot g(x)$ is fuzzy continuous function $\forall x \in J$.*

Proof. Let $x_n, x_0 \in J$, $n = 1, 2, \dots$, such that $x_n \to x_0$. Thus $D(f(x_n), f(x_0)) \to 0$, as $n \to +\infty$ and $|g(x_n) - g(x_0)| \to 0$. We need to establish that

$$\Delta_n := D(f(x_n) \odot g(x_n), f(x_0) \odot g(x_0)) \to 0,$$

as $n \to +\infty$. We have the following

$$\begin{aligned}
2\Delta_n &= D(2 \odot (f(x_n) \odot g(x_n)), 2 \odot (f(x_0) \odot g(x_0)) \\
&\quad \text{(notice for } u \in \mathbb{R}_{\mathcal{F}} \text{ that } u \oplus u = 2 \odot u) \\
&\qquad D(f(x_n) \odot g(x_n) \oplus f(x_n) \odot g(x_n) \oplus f(x_0) \odot g(x_n) \\
&\qquad \oplus f(x_n) \odot g(x_0), f(x_0) \odot g(x_n) \oplus f(x_n) \odot g(x_0) \oplus f(x_0) \\
&\qquad \odot g(x_0) \oplus f(x_0) \odot g(x_0)) \\
&\quad \le D(f(x_n) \odot g(x_n), f(x_0) \odot g(x_n)) + D(f(x_n) \odot g(x_n), f(x_n) \odot g(x_0)) \\
&\qquad + D(f(x_0) \odot g(x_n), f(x_0) \odot g(x_0)) + D(f(x_n) \odot g(x_0), f(x_0) \odot g(x_0)) \\
&\quad \text{(by Lemma 1.2)} \\
&\quad \le g(x_n) D(f(x_n), f(x_0)) + |g(x_n) - g(x_0)| D(f(x_n), \tilde{o}) \\
&\qquad + |g(x_n) - g(x_0)| D(f(x_0), \tilde{o}) + g(x_0) D(f(x_n), f(x_0)) \\
&\quad \le 2M_2 D(f(x_n), f(x_0)) + 2M_1 |g(x_n) - g(x_0)| \to 0, \quad \text{as } n \to +\infty.
\end{aligned}$$

□

We proceed with the following related result.

Theorem 2.4. *All assumptions here are as in Theorem 2.1. Define for $k \in \mathbb{Z}$, $x \in \mathbb{R}$ the fuzzy wavelet type operator*

$$(C_k f)(x) := \sum_{j=-\infty}^{\infty}{}^{*} \left(2^k \odot (FR) \int_0^{2^{-k}} f\left(t + \frac{j}{2^k}\right) dt \right) \odot \varphi(2^k x - j). \tag{9}$$

Then

$$D((C_k f)(x), f(x)) \le \omega_1^{(\mathcal{F})} \left(f, \frac{a+1}{2^k} \right), \tag{10}$$

and

$$D^*((C_k f), f) \le \omega_1^{(\mathcal{F})} \left(f, \frac{a+1}{2^k} \right), \quad \textit{all } k \in \mathbb{Z},\ x \in \mathbb{R}. \tag{11}$$

When $f \in C^U_{\mathcal{F}}(\mathbb{R})$ then as $k \to +\infty$ we get $C_k \to I$ with rates.

Proof. We need to estimate

$$
\begin{aligned}
& D((C_k f)(x), f(x)) \\
& = D\left(\sum_{\substack{j \\ 2^k x - j \in [-a,a]}}^{*} \left(2^k \odot (FR)\int_0^{2^{-k}} f\left(t + \frac{j}{2^k}\right) dt \right) \odot \varphi(2^k x - j), f(x) \odot 1 \right) \\
& = D\left(\sum_{\substack{j \\ 2^k x - j \in [-a,a]}}^{*} \left(2^k \odot (FR)\int_0^{2^{-k}} f\left(t + \frac{j}{2^k}\right) dt \right) \odot \varphi(2^k x - j), \right. \\
& \qquad \left. \sum_{\substack{j \\ 2^k x - j \in [-a,a]}}^{*} \left(2^k \odot (FR)\int_0^{2^{-k}} (f(x) \odot 1) dt \right) \odot \varphi(2^k x - j) \right) \\
& \le \sum_{\substack{j \\ 2^k x - j \in [-a,a]}} D\left(\left(\left(2^k \odot (FR)\int_0^{2^{-k}} f\left(t + \frac{j}{2^k}\right) dt \right) \odot \varphi(2^k x - j), \right.\right. \\
& \qquad \left. \left(2^k \odot (FR)\int_0^{2^{-k}} (f(x) \odot 1) dt \right) \odot \varphi(2^k x - j) \right) \\
& \le 2^k \sum_{\substack{j \\ 2^k x - j \in [-a,a]}} \varphi(2^k x - j) D\left((FR)\int_0^{2^{-k}} f\left(t + \frac{j}{2^k}\right) dt, (FR)\int_0^{2^{-k}} f(x) dt \right) \\
& \text{(by Lemma 1.6)} \\
& \le 2^k \sum_{\substack{j \\ 2^k x - j \in [-a,a]}} \varphi(2^k x - j) \int_0^{2^{-k}} D\left(f\left(t + \frac{j}{2^k}\right), f(x) \right) dt =: (*)
\end{aligned}
$$

(here $0 \le t \le \frac{1}{2^k}$ and $\left|x - \frac{j}{2^k}\right| \le \frac{a}{2^k}$, thus $\left|t + \frac{j}{2^k} - x\right| \le \frac{a+1}{2^k}$). Hence

$$
(*) \le 2^k \sum_{\substack{j \\ 2^k x - j \in [-a,a]}} \varphi(2^k x - j) \omega_1^{(\mathcal{F})}\left(f, \frac{a+1}{2^k}\right) 2^{-k} = \omega_1^{(\mathcal{F})}\left(f, \frac{a+1}{2^k}\right). \qquad \square
$$

Next we give the corresponding result for the last fuzzy wavelet type operator we are dealing with.

Theorem 2.5. *All assumptions here are as in Theorem* 2.1. *Define for* $k \in \mathbb{Z}$, $x \in \mathbb{R}$ *the fuzzy wavelet type operator*

$$(D_k f)(x) := \sum_{j=-\infty}^{\infty}{}^{*} \delta_{kj}(f) \odot \varphi(2^k x - j), \tag{12}$$

where $\delta_{kj}(f) := \sum_{\tilde{r}=0}^{n}{}^{*} w_{\tilde{r}} \odot f\left(\frac{j}{2^k} + \frac{\tilde{r}}{2^k n}\right)$, $n \in \mathbb{N}$, $w_{\tilde{r}} \geq 0$, $\sum_{\tilde{r}=0}^{n} w_{\tilde{r}} = 1$. (13)

Then
$$D((D_k f)(x), f(x)) \leq \omega_1^{(\mathcal{F})}\left(f, \frac{a+1}{2^k}\right), \tag{14}$$

and
$$D^*(D_k f, f) \leq \omega_1^{(\mathcal{F})}\left(f, \frac{a+1}{2^k}\right), \quad \textit{all } k \in \mathbb{Z},\ x \in \mathbb{R}. \tag{15}$$

When $f \in C_{\mathcal{F}}^{U}(\mathbb{R})$ *then as* $k \to +\infty$ *we get* $D_k \to I$ *with rates.*

Proof. We need to upper bound

$$D((D_k f)(x), f(x)) = D\left(\sum_{\substack{j \\ 2^k x - j \in [-a,a]}}^{*} \left(\sum_{\tilde{r}=0}^{n}{}^{*} w_{\tilde{r}} \odot f\left(\frac{j}{2^k} + \frac{\tilde{r}}{2^k n}\right)\right) \cdot \varphi(2^k x - j), \right.$$
$$\left. \sum_{\substack{j \\ 2^k x - j \in [-a,a]}}^{*} f(x) \odot \varphi(2^k x - j)\right)$$
$$\leq \sum_{\substack{j \\ 2^k x - j \in [-a,a]}} \varphi(2^k x - j) D\left(\sum_{\tilde{r}=0}^{n}{}^{*}\left(w_{\tilde{r}} \odot f\left(\frac{j}{2^k} + \frac{\tilde{r}}{2^k n}\right)\right), \sum_{\tilde{r}=0}^{n}{}^{*}(w_{\tilde{r}} \odot f(x))\right)$$
$$\leq \sum_{\substack{j \\ 2^k x - j \in [-a,a]}} \varphi(2^k x - j) \sum_{\tilde{r}=0}^{n} w_{\tilde{r}} D\left(f\left(\frac{j}{2^k} + \frac{\tilde{r}}{2^k n}\right), f(x)\right)$$
$$\left(\text{notice that } \left|\frac{j}{2^k} + \frac{\tilde{r}}{2^k n} - x\right| \leq \frac{a+1}{2^k}\right)$$
$$\leq \sum_{\substack{j \\ 2^k x - j \in [-a,a]}} \varphi(2^k x - j) \sum_{\tilde{r}=0}^{n} w_{\tilde{r}} \omega_1^{(\mathcal{F})}\left(f, \frac{a+1}{2^k}\right) = \omega_1^{(\mathcal{F})}\left(f, \frac{a+1}{2^k}\right). \qquad \square$$

Next we prove optimality for three of the above main results.

Proposition 2.6. *Inequality* (2) *is attained, that is sharp.*

Proof. Take $\varphi(x) = \chi_{[-\frac{1}{2},\frac{1}{2})}(x)$, the characteristic function on $\left[-\frac{1}{2},\frac{1}{2}\right)$. Fix $u \in \mathbb{R}_{\mathcal{F}}$ and take $f(x) = q(x) \odot u$, where

$$q(x) := \begin{cases} 0, & x \leq -2^{-k-1} \\ 1, & x \geq 0, \\ 2^{k+1}x + 1, & -2^{-k-1} < x < 0, \end{cases}$$

$k \in \mathbb{Z}$ fixed, $x \in \mathbb{R}$. Clearly $q(x) \geq 0$. We observe that

$$\begin{aligned} (B_k f)(x) &= \sum_{j=-\infty}^{\infty}{}^{*} q\left(\frac{j}{2^k}\right) \odot u \odot \varphi(2^k x - j) \\ &= \left(\sum_{j=-\infty}^{\infty} q\left(\frac{j}{2^k}\right) \varphi(2^k x - j)\right) \odot u = \left(\sum_{j=0}^{\infty} \varphi(2^k x - j)\right) \odot u. \end{aligned}$$

Hence

$$D((B_k f)(-2^{-k-1}), f(-2^{-k-1}) = D\left(\left(\sum_{j=0}^{\infty} \varphi\left(-\frac{1}{2} - j\right)\right) \odot u, \tilde{o}\right) = D(u, \tilde{o}).$$

Furthermore we see that

$$\begin{aligned} \omega_1^{(\mathcal{F})}(f, 2^{-k-1}) &= \sup_{\substack{x,y\in\mathbb{R} \\ |x-y|\leq 2^{-k-1}}} D(f(x), f(y)) = \sup_{\substack{x,y\in\mathbb{R} \\ |x-y|\leq 2^{-k-1}}} D(q(x) \odot u, q(y) \odot u) \\ &\text{(by Lemma 1.2)} \\ &\leq \left(\sup_{\substack{x,y\in\mathbb{R} \\ |x-y|\leq 2^{-k-1}}} |q(x) - q(y)|\right) D(u, \tilde{o}) = 1 \cdot D(u, \tilde{o}). \end{aligned}$$

That is, we got that

$$\omega_1^{(\mathcal{F})}(f, 2^{-k-1}) \leq D(u, \tilde{o}).$$

So that by (2) and the above we find

$$D((B_k f)(-2^{-k-1}), f(-2^{-k-1})) = \omega_1^{(\mathcal{F})}(f, 2^{-k-1}),$$

proving the sharpness of (2). ☐

Proposition 2.7. *Inequalities* (10) *and* (14) *are attained, i.e. they are sharp.*

Proof. (I) Consider as optimal elements φ, q, u, and f, exactly as in the proof of Proposition 3. Here $a = \frac{1}{2}$. We observe that

$$\begin{aligned}
\omega_1^{(\mathcal{F})}\left(f, \frac{a+1}{2^k}\right) &= \omega_1^{(\mathcal{F})}\left(f, \frac{3}{2^{k+1}}\right) = \sup_{\substack{x,y \\ |x-y|\le \frac{3}{2^{k+1}}}} D(f(x), f(y)) \\
&= \sup_{\substack{x,y \\ |x-y|\le \frac{3}{2^{k+1}}}} D(q(x)\odot u, q(y)\odot u) \\
&\text{(by Lemma 1.2)} \\
&\le \left(\sup_{\substack{x,y \\ |x-y|\le \frac{3}{2^{k+1}}}} |q(x)-q(y)|\right) D(u, \tilde{o}) \\
&= \left(\sup_{\substack{x,y \\ |x-y|\le \frac{1}{2^{k+1}}}} |q(x)-q(y)|\right) D(u, \tilde{o}) = 1 \cdot D(u, \tilde{o}).
\end{aligned}$$

That is,

$$\omega_1^{(\mathcal{F})}\left(f, \frac{a+1}{2^k}\right) \le D(u, \tilde{o}).$$

Call

$$\gamma_{kj}(f) := 2^k \odot (FR)\int_0^{2^{-k}} f\left(t + \frac{j}{2^k}\right) dt.$$

We obtain

$$\begin{aligned}
\gamma_{k(-1)}(f) &= 2^k \odot (FR)\int_0^{2^{-k}} \left(q\left(t - \frac{1}{2^k}\right)\odot u\right) dt = \left(2^k \int_0^{2^{-k}} q\left(t - \frac{1}{2^k}\right) dt\right) \odot u \\
&= \left(2^k \int_{-\frac{1}{2^k}}^0 q(t)dt\right) \odot u = \left(2^k \int_{-\frac{1}{2^{k+1}}}^0 q(t)dt\right) \odot u = \frac{1}{4} \odot u.
\end{aligned}$$

That is,

$$\gamma_{k(-1)}(f) = \frac{1}{4} \odot u.$$

Moreover $\gamma_{k(-2)}(f)=\tilde{o}$, and $\gamma_{kj}(f)=\tilde{o}$, all $j\leq -2$, and $\gamma_{kj}(f)=u$, all $j\geq 0$. Hence

$$(C_k f)(x)=\left[\frac{1}{4}\varphi(2^k x+1)+\sum_{j=0}^{+\infty}\varphi(2^k x-j)\right]\odot u.$$

We easily see then that

$$(C_k f)\left(-\frac{1}{2^{k+1}}\right)=u,\quad \text{also } f\left(-\frac{1}{2^{k+1}}\right)=\tilde{o}.$$

Therefore

$$D\left((C_k f)\left(-\frac{1}{2^{k+1}}\right),f\left(-\frac{1}{2^{k+1}}\right)\right)=D(u,\tilde{o}).$$

From the above and (10) we conclude that

$$D\left((C_k f)\left(-\frac{1}{2^{k+1}}\right),f\left(-\frac{1}{2^{k+1}}\right)\right)=\omega_1^{(\mathcal{F})}\left(f,\frac{a+1}{2^k}\right),\quad k\in\mathbb{Z},$$

proving the sharpness of (10).

(II) The sharpness of (14) is treated similarly to (I). Notice that $\delta_{kj}(f)=u$, all $j\geq 0$, and $\delta_{kj}(f)=\tilde{o}$, all $j\leq -2$. We observe that

$$\varphi\left(2^k\left(-\frac{1}{2^{k+1}}\right)-(-1)\right)=\varphi\left(\frac{1}{2}\right)=0.$$

Furthermore

$$\begin{aligned}
&D\left((D_k f)\left(-\frac{1}{2^{k+1}}\right),f\left(-\frac{1}{2^{k+1}}\right)\right)\\
&=D\left(\sum_{j=-\infty}^{\infty}{}^{*}\,\delta_{kj}(f)\odot\varphi\left(2^k\left(-\frac{1}{2^{k+1}}\right)-j\right),\tilde{o}\right)\\
&=D\left(\left(\sum_{j=0}^{\infty}1\varphi\left(-\frac{1}{2}-j\right)\right)\odot u,\tilde{o}\right)=D(1\odot u,\tilde{o})=D(u,\tilde{o}).
\end{aligned}$$

So that by (14) and the above

$$D\left((D_k f)\left(-\frac{1}{2^{k+1}}\right),f\left(-\frac{1}{2^{k+1}}\right)\right)=\omega_1^{(\mathcal{F})}\left(f,\frac{a+1}{2^k}\right),$$

proving sharpness of (14). $\square$

Remark 1. *We notice that*

$$(L_k f)(x) = L_0(f(2^{-k}\cdot))(2^k x), \quad \text{all } x \in \mathbb{R},\ k \in \mathbb{Z},$$

where $L_k = B_k,\ A_k,\ C_k,\ D_k$. Clearly L_k's are linear over $\mathbb{R}$ operators.

In the following we present a monotonicity result for the fuzzy wavelet type operators B_k and D_k. For that we need

Definition 2.8. Let $f\colon \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. Then f is called a *nondecreasing* function iff whenever $x_1 \leq x_2$, $x_1, x_2 \in \mathbb{R}$, we have that $f(x_1) \leq f(x_2)$, i.e. $(f(x_1))_-^{(r)} \leq (f(x_2))_-^{(r)}$ and $(f(x_1))_+^{(r)} \leq (f(x_2))_+^{(r)}$, $\forall r \in [0,1]$.

Theorem 2.9. *Let $f \in C(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, and the scaling function $\varphi(x)$ a real valued bounded function with supp $\varphi \subseteq [-a, a]$, $0 < a < +\infty$, such that*

(i) $\sum\limits_{j=-\infty}^{\infty} \varphi(x-j) \equiv 1$ *on* $\mathbb{R}$,

(ii) *there exists a $b \in \mathbb{R}$ such that φ is nondecreasing for $x \leq b$ and φ is nonincreasing for $x \geq b$,*

(*the above imply $\varphi \geq 0$*)*. Let $f(x)$ be nondecreasing fuzzy function. Then $(B_k f)(x)$, $(D_k f)(x)$ are nondecreasing fuzzy valued functions for any $k \in \mathbb{Z}$.*

Remark 2. *We give two examples of φ's as in Theorem* 2.9.

(i)

$$\varphi(x) = \begin{cases} 1, & -\frac{1}{2} \leq x < \frac{1}{2}, \\ 0, & \textit{elsewhere.} \end{cases}$$

(ii)

$$\varphi(x) = \begin{cases} x+1, & -1 \leq x \leq 0, \\ 1-x, & 0 < x \leq 1, \\ 0, & \textit{elsewhere.} \end{cases}$$

Proof of Theorem 2.9. Let $x_n, x \in \mathbb{R}$ such that $x_n \to x$, as $n \to +\infty$. Then $D(f(x_n), f(x)) \to 0$ by fuzzy continuity of f. But we have

$$D(f(x_n), f(x)) = \sup_{r \in [0,1]} \max\{|(f(x_n))_-^{(r)} - (f(x))_-^{(r)}|, |(f(x_n))_+^{(r)} - (f(x))_+^{(r)}|\}.$$

That is, $|(f(x_n))_\pm^{(r)} - (f(x))_\pm^{(r)}| \to 0$, all $0 \leq r \leq 1$, as $n \to +\infty$, respectively. Therefore $(f)_\pm^{(r)} \in C(\mathbb{R}, \mathbb{R})$, all $0 \leq r \leq 1$, i.e. real valued continuous functions on $\mathbb{R}$. Since f is fuzzy nondecreasing by Definition 2.8, we get that $(f)_{\pm}^{(r)}$ are nondecreasing, $\forall r \in [0,1]$, respectively. Then by Theorem 6.3, p. 156, [2], see also [5], we get that the corresponding real wavelet type operators map to the functions $(B_k(f)_{\pm}^{(r)})(x)$ that are nondecreasing on $\mathbb{R}$ for all $r \in [0,1]$, any $k \in \mathbb{Z}$. Also by Lemma 8.2, p. 186, [2], see also [1], we get that the corresponding real wavelet type operators map to the functions $(D_k(f)_{\pm}^{(r)})(x)$ that are nondecreasing on $\mathbb{R}$ for all $r \in [0,1]$, any $k \in \mathbb{Z}$. We notice for any $r \in [0,1]$ that

$$[(B_k f)(x)]^r = \sum_{j=-\infty}^{+\infty} \left[f\left(\frac{j}{2^k}\right)\right]^r \varphi(2^k x - j).$$

That is

$$\begin{aligned}
&\left[((B_k f)(x))_-^{(r)}, ((B_k f)(x))_+^{(r)}\right] \\
&= \sum_{j=-\infty}^{+\infty} \left[\left(f\left(\frac{j}{2^k}\right)\right)_-^{(r)}, \left(f\left(\frac{j}{2^k}\right)\right)_+^{(r)}\right] \varphi(2^k x - j) \\
&= \left[\sum_{j=-\infty}^{+\infty} \left(f\left(\frac{j}{2^k}\right)\right)_-^{(r)} \varphi(2^k x - j), \sum_{j=-\infty}^{+\infty} \left(f\left(\frac{j}{2^k}\right)\right)_+^{(r)} \varphi(2^k x - j)\right] \\
&= \left[(B_k(f)_-^{(r)})(x), (B_k(f)_+^{(r)})(x)\right].
\end{aligned}$$

So whenever $x_1 \le x_2$ we get $(f)_{\pm}^{(r)}(x_1) \le (f)_{\pm}^{(r)}(x_2)$, respectively, and

$$(B_k(f)_{\pm}^{(r)})(x_1) \le (B_k(f)_{\pm}^{(r)})(x_2), \quad \forall r \in [0,1].$$

Therefore $(B_k f)(x_1) \le (B_k f)(x_2)$, that is $(B_k f)$ is nondecreasing.

Next we observe that

$$[(D_k f)(x)]^r = \sum_{j=-\infty}^{+\infty} \left(\sum_{\tilde{r}=0}^{n} w_{\tilde{r}} \left[f\left(\frac{j}{2^k} + \frac{\tilde{r}}{2^k n}\right)\right]^r\right) \varphi(2^k x - j).$$

That is

$$\begin{aligned}
&\left[((D_k f)(x))_-^{(r)}, ((D_k f)(x))_+^{(r)}\right] \\
&= \sum_{j=-\infty}^{+\infty} \left(\sum_{\tilde{r}=0}^{n} w_{\tilde{r}} \left[\left(f\left(\frac{j}{2^k} + \frac{\tilde{r}}{2^k n}\right)\right)_-^{(r)}, \left(f\left(\frac{j}{2^k} + \frac{\tilde{r}}{2^k n}\right)\right)_+^{(r)}\right]\right) \varphi(2^k x - j)
\end{aligned}$$

$$= \left[\sum_{j=-\infty}^{+\infty} \left(\sum_{\tilde{r}=0}^{n} w_{\tilde{r}} \left(f\left(\frac{j}{2^k} + \frac{\tilde{r}}{2^k n} \right) \right)_{-}^{(r)} \varphi(2^k x - j), \right. \right.$$

$$\left. \sum_{j=-\infty}^{+\infty} \left(\sum_{\tilde{r}=0}^{n} w_{\tilde{r}} \left(f\left(\frac{j}{2^k} + \frac{\tilde{r}}{2^k n} \right) \right)_{+}^{(r)} \right) \varphi(2^k x - j) \right]$$

$$= \left[(D_k(f)_{-}^{(r)})(x), (D_k(f)_{+}^{(r)})(x) \right].$$

So whenever $x_1 \le x_2$ we get

$$(D_k(f)_{\pm}^{(r)})(x_1) \le (D_k(f)_{\pm}^{(r)})(x_2), \quad \forall r \in [0,1].$$

Therefore $(D_k f)(x_1) \le (D_k f)(x_2)$, so that $(D_k f)$ is nondecreasing. $\square$

Finally we present the corresponding monotonicity results for the fuzzy wavelet type operators A_k, C_k.

Theorem 2.10. *Let $f \in C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ and φ as in Theorem 2.9 which is continuous on $[-a, a]$. Let $f(x)$ be nondecreasing fuzzy function. Then $(A_k f)(x)$ is a nondecreasing fuzzy valued function for any $k \in \mathbb{Z}$.*

Proof. Since f is fuzzy nondecreasing we get again that $(f)_{\pm}^{(r)}$ are nondecreasing, $\forall r \in [0,1]$, respectively. Then by Theorem 6.1, p. 149, [2], see also [5], we get that the corresponding real wavelet type operators map to the functions $(A_k(f)_{\pm}^{(r)})(x)$ that are nondecreasing on $\mathbb{R}$ for all $r \in [0,1]$, any $k \in \mathbb{Z}$.

Using Theorem 1.5, for any $r \in [0,1]$ we notice that

$$[\langle f, \varphi_{kj} \rangle]^r = \left[\int_{\frac{j-a}{2^k}}^{\frac{j+a}{2^k}} (f(t) \odot \varphi_{kj}(t))_{-}^{(r)} dt, \int_{\frac{j-a}{2^k}}^{\frac{j+a}{2^k}} (f(t) \odot \varphi_{kj}(t))_{+}^{(r)} dt \right]$$

$$= \left[\int_{\frac{j-a}{2^k}}^{\frac{j+a}{2^k}} (f(t))_{-}^{(r)} \varphi_{kj}(t) dt, \int_{\frac{j-a}{2^k}}^{\frac{j+a}{2^k}} (f(t))_{+}^{(r)} \varphi_{kj}(t) dt \right].$$

We observe for any $r \in [0,1]$ that

$$[(A_k f)(x)]^r = \sum_{j=-\infty}^{+\infty} [\langle f, \varphi_{kj} \rangle]^r \varphi_{kj}(x).$$

That is

$$\left[((A_k f)(x))_{-}^{(r)}, ((A_k f)(x))_{+}^{(r)} \right]$$

$$= \sum_{j=-\infty}^{+\infty} \left[\int_{\frac{j-a}{2^k}}^{\frac{j+a}{2^k}} (f(t))_{-}^{(r)} \varphi_{kj}(dt) dt, \int_{\frac{j-a}{2^k}}^{\frac{j+a}{2^k}} (f(t))_{+}^{(r)} \varphi_{kj}(t) dt \right] \varphi_{kj}(x)$$

$$= \left[\sum_{j=-\infty}^{+\infty} \left(\int_{\frac{j-a}{2^k}}^{\frac{j+a}{2^k}} (f(t))_-^{(r)} \varphi_{kj}(t) dt \right) \varphi_{kj}(x), \sum_{j=-\infty}^{+\infty} \left(\int_{\frac{j-a}{2^k}}^{\frac{j+a}{2^k}} (f(t))_+^{(r)} \varphi_{kj}(t) dt \right) \varphi_{kj}(x) \right]$$
$$= \left[(A_k(f)_-^{(r)})(x), (A_k(f)_+^{(r)})(x) \right].$$

So whenever $x_1 \leq x_2$ we have that $(f)_\pm^{(r)}(x_1) \leq (f)_\pm^{(r)}(x_2)$, respectively, and

$$(A_k(f)_\pm^{(r)})(x_1) \leq (A_k(f)_\pm^{(r)})(x_2), \quad \forall r \in [0,1].$$

Hence $(A_k f)(x_1) \leq (A_k f)(x_2)$, that is $(A_k f)$ is nondecreasing. □

Theorem 2.11. *Let f and φ as in Theorem 2.9. Let $f(x)$ be nondecreasing fuzzy function. Then $(C_k f)(x)$ is a nondecreasing fuzzy valued function for any $k \in \mathbb{Z}$.*

Proof. By Lemma 8.2, p. 186, [2], see also [1], we get that the corresponding real wavelet type operators map to the functions $(C_k(f)_\pm^{(r)})(x)$ that are nondecreasing on $\mathbb{R}$ for all $r \in [0,1]$, any $k \in \mathbb{Z}$. Using Theorem 1.5, for any $r \in [0,1]$ we notice that

$$\begin{aligned}
[(C_k f)(x)]^r &= \sum_{j=-\infty}^{+\infty} \left[2^k \odot (FR) \int_0^{2^{-k}} f\left(t + \frac{j}{2^k}\right) dt \right]^r \varphi(2^k x - j) \\
&= \sum_{j=-\infty}^{+\infty} \left[2^k \odot (FR) \int_{2^{-k}j}^{2^{-k}(j+1)} f(t) dt \right]^r \varphi(2^k x - j) \\
&= \sum_{j=-\infty}^{+\infty} \left[2^k \int_{2^{-k}j}^{2^{-k}(j+1)} (f)_-^{(r)}(t) dt, 2^k \int_{2^{-k}j}^{2^{-k}(j+1)} (f)_+^{(r)}(t) dt \right] \varphi(2^k x - j) \\
&= \left[\sum_{j=-\infty}^{+\infty} \left(2^k \int_{2^{-k}j}^{2^{-k}(j+1)} (f)_-^{(r)}(t) dt \right) \varphi(2^k x - j), \right. \\
&\qquad \left. \sum_{j=-\infty}^{+\infty} \left(2^k \int_{2^{-k}j}^{2^{-k}(j+1)} (f)_+^{(r)}(t) dt \right) \varphi(2^k x - j) \right] \\
&= \left[(C_k(f)_-^{(r)})(x), (C_k(f)_+^{(r)})(x) \right].
\end{aligned}$$

That is, for any $r \in [0,1]$ we found

$$\left[((C_k f)(x))_-^{(r)}, ((C_k f)(x))_+^{(r)} \right] = \left[(C_k(f)_-^{(r)})(x), (C_k(f)_+^{(r)})(x) \right].$$

So whenever $x_1 \leq x_2$ we have $(f)_{\pm}^{(r)}(x_1) \leq (f)_{\pm}^{(r)}(x_2)$ and

$$(C_k(f)_{\pm}^{(r)})(x_1) \leq (C_k(f)_{\pm}^{(r)})(x_2), \quad \forall r \in [0,1],$$

respectively. Hence $(C_k f)(x_1) \leq (C_k f)(x_2)$, that is $(C_k f)$ is nondecreasing. $\square$

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