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A NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF POSITIVE SOLUTIONS TO A CLASS OF SINGULAR SECOND-ORDER BOUNDARY VALUE PROBLEMS

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Abstract. We study the existence of positive solutions to a class of second order singular boundary value problems by means of lower and upper solutions. A necessary and sufficient condition for the existence of $C[0,1]$ and $C^1[0,1]$ positive solutions is obtained. Results of this paper extend and include some results in [4-7].

1. Introduction

Boundary value problems concerning the generalized Emden-Fowler equations

$$x''(t) + a(t)x^{\lambda}(t) = 0, \quad t \in (0,1), \tag{1}$$

where $a(t) \in C(0,1)$, $a(t) \geq 0$ for $t \in (0,1)$, has been profusely studied by many authors. Its origin lies in theories concerning gaseous dynamics in astrophysics around the turn of the century [1]. See [5] for more recent applications. When $a(t) \in C[0,1]$, problem (1) is nonsingular. One can find classical results about the existence of this case in [5]. When $a(t)$ is not continuous at the end points of (0,1) (including the case that $a(t)$ is unbounded on (0,1), or $\lambda < 0$, problem (1) is singular. Also, much attention has been paid to the existence of positive solutions, as can be seen in [2-7]. When $\lambda > 0$, Wei

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[6] and Zhang [8] studied problem (1). When $\lambda < 0$, Fink-Gatic-Hernadea-waltman [2], Luning-Perry [3], Taliaferro [4], Wong [5], and wei [7] researched problem (1). Results of these papers are significant.

However, by general analyzing, we can find that, in these papers, the argument relies on the fact that $f(x) = x^{\lambda}$ is monotones. It's worth indicating that the monotonicity condition is key for these papers. For example, the existence of positive solutions for the case $\lambda < 0$ has been studied completely by Taliaferro [4] with the shooting method. But when $\lambda > 0$ or replace $a(t)x^{\lambda}(t)$ with $a(t)x^{p}(t) + b(t)x^{-q}, p$ and $q > 0$ in problem (1), the monotonicity on x asked by the shooting method is not holds. This makes the method unless.

In this paper we study the singular boundary value problems

$$\begin{cases} x''(t) + a(t)x^{p}(t) + b(t)x^{-q} = 0, \quad t \in (0,1), \\ x(0) = x(1) = 0, \end{cases} \tag{2}$$

where $q > 0, 0 < p < 1, a, b \in C((0,1),[0,\infty))$ and $a(t), b(t)$ may be singular at the end points of (0,1). A necessary and sufficient condition for the existence of $C[0,1]$ and $C^{1}[0,1]$ positive solutions of problem (2) is obtained by using the method of lower and upper solutions. Main results of this paper extend and include those of [4,6,7]. When $a(t)$ and $b(t)$ are continuous on $[0,1]$, main results of this paper are new also.

2. Preliminaries

We shall give some preliminary considerations and some lemmas. In our discussion, by a positive solution of (2) we mean a function $x(t) \in C[0,1] \cap C^{2}(0,1)$ which satisfies boundary value problem (2) and $x(t) > 0$ holds for $t \in (0,1)$. If in addition there is a solution $x(t) \in C^{1}[0,1]$, i.e., both $x'(0^{+})$ and $x'(1^{-})$ exist, we call it a $C^{1}[0,1]$ solution.

For the sake of convenience, we list the hypothesis as follows, which is assumed throughout the paper.

$$(H): \quad a(t), b(t) \in C((0,1),[0,\infty)), \quad q > 0, \quad 0 < p < 1.$$

The following lemma is important for this paper.

Lemma 1. *[8]. If the following boundary value problem*

$$\begin{cases} x''(t) + f(t,x) = 0, \quad t \in (0,1), \\ x(0) = r_1, x(1) = r_2, \end{cases}$$

where $f(t,x) \in C((0,1) \times I, R), r_1, r_2 \in I$, $I \subset R$, *has lower and upper solutions* $\alpha(t), \beta(t)$, *i.e.,* $\alpha(t), \beta(t) \in C[0,1] \cap C^2(0,1)$, $\alpha(t) \le \beta(t)$, $t \in [0,1]$, $\alpha(0) = \beta(0) = r_1, \alpha(1) = \beta(1) = r_2$, *then it possesses at least one solution* $x(t) \in C[0,1]$ *such that* $\alpha(t) \le x(t) \le \beta(t)$ *for* $t \in [0,1]$.

3. Main results

Now we state the main results of this paper as follows.

Theorem 1. *Suppose condition* (H) *and* $q \le p$ *hold. Then a necessary and sufficient condition for problem* (2) *to have* $C[0,1]$ *positive solutions is that*

$$0 < \int_0^1 (a(t) + b(t))dt < \infty. \tag{3}$$

Proof. **Necessity.** Assume $x \in C[0,1]$ is a positive solution of problem (2). Then from the boundary value condition there exists $t_0 \in (0,1)$ such that $x'(t_0) = 0$. Then

$$\int_{t_0}^t [a(s)x^p(s) + b(s)x^{-q}(s)]ds = -\int_{t_0}^t x''(s)ds = -x'(t), \quad t \in (0,1). \tag{4}$$

Multiplying both sides of (4) by $x^{q-p}(t)$ and then integrating on $[t_0, 1]$, we get

$$0 \le \int_{t_0}^1 x^{q-p}(t) \int_{t_0}^t [a(s)x^p(s) + b(s)x^{-q}(s)]dsdt = \int_{t_0}^1 -x^{q-p}(t)x'(t)dt < \infty. \tag{5}$$

From $x''(t) \le 0$, we know $x'(t) \le 0$ for $t \in [t_0, 1)$. So $x(t)$ is decreasing on $[t_0, 1)$. Thus $x^{-q}(s)$ is increasing on $[t_0, t]$ for $t \in [t_0, 1)$, $x^{p-q}(s)$ and $x^p(s)$ are decreasing on $[t_0, t]$ for $t \in [t_0, 1)$. This implies

$$\int_{t_0}^t x^{p-q}(t)[a(s) + b(s)]ds \le \int_{t_0}^t x^{p-q}(s)[a(s) + b(s)]ds, t \in [t_0, 1).$$

Thus we have,

$$\int_{t_0}^t [a(s) + b(s)]ds \le x^{q-p}(t) \int_{t_0}^t x^{p-q}(s)[a(s) + b(s)]ds, \quad t \in [t_0, 1). \tag{6}$$

We have, from (4)-(6),

$$
\begin{aligned}
0 &\le \int_{t_0}^{1}(1-s)[a(s)+b(s)]ds \\
&= \int_{t_0}^{1}\int_{t_0}^{t}[a(s)+b(s)]dsdt \\
&\le \int_{t_0}^{1}\int_{t_0}^{t} x^{q-p}(t)x^{p-q}(s)[a(s)+b(s)]dsdt \\
&\le \int_{t_0}^{1}\int_{t_0}^{t} x^{-p}(t)\frac{x^{q}(t)}{x^{q}(s)}a(s)x^{p}(s)dsdt + \int_{t_0}^{1}\int_{t_0}^{t} x^{p}(s)x^{q-p}(t)b(s)x^{-q}(s)dsdt \\
&= \int_{t_0}^{1} -x^{-p}(t)x'(t)dt + \int_{t_0}^{1} -x^{p}(t_0)x^{q-p}(t)x'(t)dt < \infty.
\end{aligned}
\tag{7}
$$

Similarly, we can find that

$$
0 \le \int_{0}^{t_0} s[a(s)+b(s)]ds \le \frac{1}{1-p}x^{1-p}(t_0) + \frac{1}{1+q-p}x^{1+q}(t_0) < \infty. \tag{8}
$$

From (7) and (8), we show that

$$
\int_0^1 [a(t)+b(t)]dt < \infty. \tag{9}
$$

Moreover, if $\int_0^1[a(t)+b(t)]dt \equiv 0$, then $a(t)+b(t)\equiv 0$ holds on (0,1), i.e., $a(t)\equiv -b(t)$ holds on (0,1). Noting that $a(t)\ge 0, b(t)\ge 0$ for $t\in(0,1)$, thus $a(t)\equiv b(t)\equiv 0$ holds on (0,1), thus it is impossible for the problem (2) to have nontrivial solution. Hence

$$
0 < \int_0^1 [(a(t)+b(t)]dt < \infty. \tag{10}
$$

This ends the proof of necessity.

Sufficiency. Suppose that (3) holds. From Lemma 1 and Lemma 2 of [8] we know that there exists $g(t)$ such that

$$
g(t)\in C[0,1]\cap C^2(0,1), \quad g(t)>0, \quad g''(t)\le 0, \quad t\in(0,1), \quad g(0)=g(1)=0,
$$

$$
\int_0^1 t(1-t)g^{-q-p}(t)[a(t)+b(t)]dt < \infty. \tag{11}
$$

Furthermore, it is easy to verify that

$$c^{-q}a(t) + c^{p}b(t) \le c^{-q}a(t) + c^{-q}b(t), \quad 0 \le c \le 1, \quad t \in (0,1), \tag{12}$$

$$c^{p}a(t) + c^{-q}b(t) \le c^{p}a(t) + c^{p}b(t), \quad c \ge 1, \quad t \in (0,1). \tag{13}$$

When $g(t) < 1, t \in (0,1)$, then from (11) and (12) we know that

$$\int_0^1 t(1-t)g^{-p}(t)[a(t)g^{p}(t)+b(t)g^{-q}(t)]dt \le \int_0^1 t(1-t)g^{-q-p}(t)[a(t)+b(t)]dt < \infty. \tag{14}$$

When $g(t) \ge 1, t \in (0,1)$, then from (3) and (13) we know that

$$\int_0^1 t(1-t)g^{-p}(t)[a(t)g^{p}(t)+b(t)g^{-q}(t)]dt \le \int_0^1 t(1-t)g^{-p}(t)g^{p}(t)[a(t)+b(t)]dt < \infty. \tag{15}$$

From (14) and (15), we know that

$$\int_0^1 t(1-t)g^{-p}(t)[a(t)g^{p}(t) + b(t)g^{-q}(t)]dt < \infty, \quad t \in (0,1). \tag{16}$$

Let

$$R_1(t) = (1-t)\int_0^t s^{1+p}(1-s)^{p}[a(s)+b(s)]ds + t\int_t^1 s^{p}(1-s)^{1+p}[a(s)+b(s)]ds,$$

$$R_2(t) = g(t) + (1-t)\int_0^t sg^{-p}(s)[a(s)g^{p}(s) + b(s)g^{-q}(s)]ds$$
$$+ t\int_t^1 (1-s)g^{-p}(s)[a(s)g^{p}(s) + b(s)g^{-q}(s)]ds.$$

Then obviously,

$$R_i \in C[0,1] \cap C^2(0,1), \quad R_i(0) = R_i(1) = 0, \quad i = 1,2,$$

$$R_1^{''}(t) = -t^{p}(1-t)^{p}[a(t)+b(t)], \ t \in (0,1)$$

$$R_2^{''}(t) \le -g^{-p}(t)[a(t)g^{p}(t) + b(t)g^{-q}(t)], \ t \in (0,1).$$

If we let

$$L_1 = \int_0^1 s^{1+p}(1-s)^{1+p}[a(s)+b(s)]ds,$$

$$L_2 = \int_0^1 s(1-s)g^{-p}(s)[a(s)g^{p}(s) + b(s)g^{-q}(s)]ds + \max_{t\in[0,1]} g(t).$$

Then

$$L_1 t(1-t) \le R_1(t) \le L_1, g(t) \le R_2(t) \le L_2, \quad t \in [0,1]. \tag{17}$$

Under such circumstance, we let

$$\alpha(t) = M_1 R_1(t), \quad \beta(t) = M_2 R_2(t), \quad t \in [0,1],$$

where M_1, M_2 are constants which will be given in (21). We choose constants C_1, C_2 such that $C_1 L_1 \le 1, \frac{1}{C_1} \ge 1, C_2 \ge 1$, then from (12) and (13) we get

$$\begin{aligned} a(t)\alpha^p(t) + b(t)\alpha^{-q}(t) &\ge (\frac{1}{C_1})^{-q}[a(t)C_1^p\alpha^p(t) + b(t)C_1^{-q}(t)\alpha^{-q}(t)] \\ &\ge C_1^{p+q}\alpha^p(t)[a(t)+b(t)] \\ &\ge M_1^p L_1^p C_1^{p+q} t^p(1-t)^p[a(t)+b(t)], \quad t \in (0,1), \end{aligned} \tag{18}$$

$$\begin{aligned} a(t)\beta^p(t) + b(t)\beta^{-q}(t) &\le (\frac{\beta(t)}{g(t)})^p[a(t)g^p(t) + b(t)g^{-q}(t)] \\ &\le C_2^{p+q}(\frac{\beta(t)}{g(t)})^p[a(t)g^p(t) + b(t)g^{-q}(t)] \\ &\le C_2^{p+q} M_2^p L_2^p g^{-p}(t)[a(t)g^p(t) + b(t)g^{-q}(t)], t \in (0,1). \end{aligned} \tag{19}$$

From (12) and (13) we know that there exists $K_0 > 0$ such that

$$a(t)g^p(t) + b(t)g^{-q}(t) \ge K_0 g^p(t)[a(t)+b(t)], \quad t \in (0,1).$$

Thus from the definition of R_1, R_2, for any $K > \frac{1}{K_0}$, we know that

$$R_1(t) \le K R_2(t), \quad t \in [0,1]. \tag{20}$$

Now, let

$$M_1 = \min\{1, (L_1^p C_1^{p+q})^{\frac{1}{1-p}}\}, \quad M_2 = \max\{1, \frac{1}{K_0}, (L_2^p C_2^{p+q})^{\frac{1}{1-p}}\}. \tag{21}$$

From (17),(18),(19),(20) and (21), we get

$$\begin{aligned} &\alpha''(t) + a(t)\alpha^p(t) + b(t)\alpha^{-q}(t) \\ &\ge -M_1 t^p(1-t)^p[a(t)+b(t)] + M_1^p L_1^p C_1^{p+q} t^p(1-t)^p[a(t)+b(t)] \\ &= M_1 t^p(1-t)^p[a(t)+b(t)](M_1^{p-1}L_1^p C_1^{p+q} - 1) \ge 0, \quad t \in (0,1), \end{aligned} \tag{22}$$

$$
\begin{aligned}
&\beta''(t) + a(t)\beta^p(t) + b(t)\beta^{-q}(t) \\
&\le -M_2 g^{-p}(t)[a(t)g^p(t) + b(t)g^{-q}(t)] \\
&\quad + M_2^p L_2^p C_2^{p+q} g^{-p}(t)[a(t)g^p(t) + b(t)g^{-q}(t)] \\
&= M_2 g^{-p}(t)[a(t)g^p(t) + b(t)g^{-q}(t)](M_2^{p-1} L_2^p C_2^{p+q} - 1) \le 0, \quad t \in (0,1).
\end{aligned} \tag{23}
$$

From (22) and (23), we show that $\alpha(t), \beta(t)$ are the lower and upper solutions of problem (2), respectively. Furthermore, we know

$$
0 < \alpha(t) \le \beta(t), \quad t \in (0,1), \quad \alpha(i) = \beta(i) = 0, \quad i = 0, 1.
$$

Then Lemma 1 tells us that problem (2) has a $C[0,1]$ positive solution x such that $0 < \alpha(t) \le x(t) \le \beta(t), t \in (0,1)$. This completes the proof of Theorem 1. □

Theorem 2. *Suppose condition* (H) *holds. Then a necessary and sufficient condition for problem* (2) *to have* $C^1[0,1]$ *positive solutions is that*

$$
0 < \int_0^1 (t^p(1-t)^p a(t) + t^{-q}(1-t)^{-q} b(t))dt < \infty. \tag{24}
$$

Proof. **Necessity.** Assume $x \in C^1[0,1]$ is a positive solution of problem (2). Then both $x'(0)$ and $x'(1)$ exist. Since $x''(t) \le 0, x(t) > 0, t \in (0,1), x(0) = x(1) = 0$, we can find that $x'(0) > 0 > x'(1)$. Thus there exist constant $0 < m \le M$ such that

$$
mt(1-t) \le x(t) \le Mt(1-t), \quad t \in [0,1]. \tag{25}
$$

From (24) and (25), noting that $t^p(1-t)^p a(t) + t^{-q}(1-t)^{-q} b(t) \ge 0$ and $t^p(1-t)^p a(t) + t^{-q}(1-t)^{-q} b(t) \not\equiv 0, t \in (0,1)$, we get

$$
\begin{aligned}
0 &< \int_0^1 (t^p(1-t)^p a(t) + t^{-q}(1-t)^{-q} b(t))dt \\
&\le [m^{-p} a(t) x^p(t) + M^q x^{-q}(t) b(t)]dt \\
&\le \max\{m^{-p}, M^q\} \int_0^1 [a(t)x^p(t) + b(t)x^{-q}(t)] \\
&= (x'(0) - x'(1)) \max\{m^{-p}, M^q\} < \infty.
\end{aligned}
$$

This implies (24).

Sufficiency. Suppose that (24) holds. Let

$$R(t) = (1-t)\int_0^t s[a(s)(s-s^2)^p + b(s)(s-s^2)^{-q}]ds + t\int_t^1 (1-s)[a(s)(s-s^2)^p + b(s)(s-s^2)^{-q}]ds.$$

It is easy to see that

$$R(t) \in C[0,1] \cap C^2(0,1), R''(t) = -[a(t)t^p(1-t)^p + b(t)t^{-q}(1-t)^{-q}]. \quad (26)$$

If we let

$$I_1 = \int_0^1 t(1-t)[a(t)t^p(1-t)^p + b(t)t^{-q}(1-t)^{-q}]dt,$$
$$I_2 = \int_0^1 [a(t)t^p(1-t)^p + b(t)t^{-q}(1-t)^{-q}]dt.$$

Then

$$I_1 t(1-t) \le R(t) \le I_2 t(1-t), \quad t \in [0,1]. \quad (27)$$

Let

$$\alpha(t) = m_1 R(t), \quad \beta(t) = m_2 R(t),$$

where

$$m_1 = \min\{1, I_1^{\frac{p}{1-p}}, I_2^{\frac{-q}{1+q}}\}, \quad m_2 = \max\{1, I_2^{\frac{p}{1-p}}, I_1^{\frac{-q}{1+q}}\}.$$

Then from (27), we can find that for any $t \in (0,1)$,

$$\begin{aligned}
&\alpha''(t) + a(t)\alpha^p(t) + b(t)\alpha^{-q}(t)\\
&\ge -m_1[a(t)t^p(1-t)^p) + b(t)t^{-q}(1-t)^{-q}] + m_1^p a(t)R^p(t) + b(t)m_1^{-q}R^{-q}(t)\\
&= -m_1[a(t)t^p(1-t)^p + b(t)t^{-q}(1-t)^{-q}] + m_1^p a(t)I_1^p t^p(1-t)^p\\
&\quad + m_1^{-q}b(t)I_2^{-q}t^{-q}(1-t)^{-q} \qquad (28)\\
&= m_1 a(t)t^p(1-t)^p(m_1^{p-1}I_1^p - 1) + m_1 b(t)t^{-q}(1-t)^{-q}(m_1^{-q-1}I_2^{-q} - 1) \ge 0,
\end{aligned}$$

$$\begin{aligned}
&\beta''(t) + a(t)\beta^p(t) + b(t)\beta^{-q}(t)\\
&\le -m_2[a(t)t^p(1-t)^p + b(t)t^{-q}(1-t)^{-q}]\\
&\quad + m_2^p a(t)I_2^p t^p(1-t)^p + m_2^{-q}b(t)I_1^{-q}t^{-q}(1-t)^{-q} \qquad (29)\\
&= m_2 a(t)t^p(1-t)^p(m_2^{p-1}I_2^p - 1) + m_2 b(t)t^{-q}(1-t)^{-q}(m_2^{-q-1}I_1^{-q} - 1) \le 0.
\end{aligned}$$

From (28) and (29) show that $\alpha(t), \beta(t)$ are the lower and upper solutions of problem (2) respectively, furthermore, we know that

$$0 < \alpha(t) \leq \beta(t), \quad t \in (0,1), \quad \alpha(i) = \beta(i) = 0, \quad i = 0, 1.$$

Thus from Lemma 1 we may draw a conclusion that problem (2) have a $C[0,1]$ positive solution x_* such that

$$0 < \alpha(t) \leq x_*(t) \leq \beta(t), \quad t \in (0,1). \tag{30}$$

Let's prove that the positive solution x_* is $C^1[0,1]$ solution. From (27) and (30), we know

$$\begin{aligned} 0 &\leq a(t)x_*^p(t) + b(t)x_*^{-q}(t) \\ &\leq a(t)\beta^p(t) + b(t)\alpha^{-q}(t) \\ &\leq a(t)t^p(1-t)^p m_2^p I_2^p + b(t)t^{-q}(1-t)^{-q} m_1^{-q} I_1^{-q} \\ &\leq \max\{m_1^{-q} I_1^{-q}, m_2^p I_2^p\}[a(t)t^p(1-t)^p + b(t)t^{-q}(1-t)^{-q}] < \infty, \quad t \in (0,1). \end{aligned}$$

The above inequality and (24) imply that $x_*''(t)$ is absolutely integrable over $(0,1)$. So both $x_*'(0^+)$ and $x_*'(1^-)$ are exist, i.e., $x_*'(t) \in C^1[0,1]$. This completes the proof. □

Remark (1). Theorem 1 of [6] is the special case for $b(t) \equiv 0, t \in (0,1)$ of Theorem 1 of this paper.
(2). Main results of [4] and Theorem (A) of [6] are the special cases for $b(t) \equiv 0$, $t \in (0,1)$ of Theorem 2 of this paper.
(3). If we let $a(t) \equiv 0$, $t \in (0,1)$, then from Theorem 1 and Theorem 2 of this paper we can obtain Theorem 1(IV) and Theorem 2(C) of [7].
(4). If both $a(t) \not\equiv 0$ and $b(t) \not\equiv 0$ hold, then main results in this paper is not easy to obtain for other papers.
(5). Even if we let $b(t) \equiv 0, t \in (0,1)$, $a(t)$ is continuous on $[0,1]$, main results of this paper are new also.

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