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ISOMORPHISMS BETWEEN BANACH ALGEBRAS

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Abstract. It is shown that a bijective approximate isomorphism $f : \mathcal{B} \to \mathcal{C}$ of a unital Banach algebra $\mathcal{B}$ to a unital Banach algebra $\mathcal{C}$ is an algebra isomorphism. Moreover, we prove that a bijective approximate $*$-isomorphism $f : \mathcal{B} \to \mathcal{C}$ of a unital C^*-algebra $\mathcal{B}$ to a unital C^*-algebra $\mathcal{C}$ is an algebra $*$-isomorphism, and that a bijective approximate $*$-isomorphism $f : \mathcal{B} \to \mathcal{C}$ of a unital JB^*-algebra $\mathcal{B}$ to a unital JB^*-algebra $\mathcal{C}$ is an algebra $*$-isomorphism.

1. Introduction

Our knowledge concerning the continuity properties of epimorphisms on Banach algebras, Jordan-Banach algebras, and, more generally, nonassociative complete normed algebras, is now fairly complete and satisfactory (see [7] and [8]). A basic continuity problem consists in determining algebraic conditions on a Banach algebra A which ensure that derivations on A are continuous. In 1996, Villena [8] proved that derivations on semisimple Jordan-Banach algebras are continuous.

Let E_1 and E_2 be Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$. Rassias [6] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

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for all $x \in E_1$. Găvruta [2] generalized the Rassias' result, and Park [5] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra.

Throughout this paper, let $\mathcal{B}$ be a unital Banach algebra with unit e and norm $\|\cdot\|$, and $\mathcal{C}$ a unital Banach algebra with unit e' and norm $\|\cdot\|$.

In this paper, we prove that a bijective approximate isomorphism $f : \mathcal{B} \to \mathcal{C}$ of a unital Banach algebra $\mathcal{B}$ to a unital Banach algebra $\mathcal{C}$ is an algebra isomorphism. This result is applied to unital C^*-algebras and unital JB^*-algebras.

2. Stability of isomorphisms between unital Banach algebras

We are going to show the generalized Hyers-Ulam-Rassias stability of isomorphisms between unital Banach algebras.

Theorem 1. *Let $f : \mathcal{B} \to \mathcal{C}$ be a bijective mapping satisfying $f(xy) = f(x)f(y)$ and $f(0) = 0$ for which there exists a function $\varphi : \mathcal{B} \times \mathcal{B} \to [0, \infty)$ such that*

(i) $$\sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) < \infty,$$

(ii) $$\|D_\mu f(x, y)\| := \|f(\mu x + \mu y) - \mu f(x) - \mu f(y)\| \leq \varphi(x, y)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid \ |\lambda| = 1\}$ and all $x, y \in \mathcal{B}$. Assume that (iii) $\lim_{n\to\infty} \frac{f(2^n e)}{2^n}$ *is invertible. Then the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra isomorphism.*

Proof. Put $\mu = 1 \in \mathbb{T}^1$. Replacing y by x in (ii), we get

$$\|f(2x) - 2f(x)\| \leq \varphi(x, x)$$

for all $x \in \mathcal{B}$. So one can obtain that

$$\|f(x) - \frac{1}{2}f(2x)\| \leq \frac{1}{2}\varphi(x, x),$$

and hence

$$\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^{n+1}}f(2^{n+1}x)\| \leq \frac{1}{2^{n+1}}\varphi(2^n x, 2^n x)$$

for all $x \in \mathcal{B}$. So we get

$$\|f(x) - \frac{1}{2^n}f(2^n x)\| \leq \frac{1}{2}\sum_{l=0}^{n-1}\frac{1}{2^l}\varphi(2^l x, 2^l x) \tag{1}$$

for all $x \in \mathcal{B}$.

Let x be an element in $\mathcal{B}$. For positive integers n and m with $n > m$,

$$\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x)\| \le \frac{1}{2}\sum_{l=m}^{n-1}\frac{1}{2^l}\varphi(2^l x, 2^l x),$$

which tends to zero as $m \to \infty$ by (i). So $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in \mathcal{B}$. Since $\mathcal{C}$ is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges for all $x \in \mathcal{B}$. We can define a mapping $\Theta : \mathcal{B} \to \mathcal{C}$ by

$$\Theta(x) = \lim_{n\to\infty}\frac{1}{2^n}f(2^n x) \tag{2}$$

for all $x \in \mathcal{B}$.

By (i) and (2), we get

$$\|D_1\Theta(x,y)\| = \lim_{n\to\infty}\frac{1}{2^n}\|D_1 f(2^n x, 2^n y)\| \le \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n x, 2^n y) = 0$$

for all $x, y \in \mathcal{B}$. Hence $D_1\Theta(x,y) = 0$ for all $x, y \in \mathcal{B}$. So one obtains that Θ is additive. Moreover, by passing to the limit in (1) as $n \to \infty$, we get the inequality

$$\text{(iv)} \qquad \|f(x) - \Theta(x)\| \le \frac{1}{2}\sum_{j=0}^{\infty}2^{-j}\varphi(2^j x, 2^j x)$$

for all $x \in \mathcal{B}$.

By the assumption, for each $\mu \in \mathbb{T}^1$,

$$\|f(2^n\mu x) - 2\mu f(2^{n-1}x)\| \le \varphi(2^{n-1}x, 2^{n-1}x)$$

for all $x \in \mathcal{B}$. And one can show that

$$\|\mu f(2^n x) - 2\mu f(2^{n-1}x)\| \le |\mu|\cdot\|f(2^n x) - 2f(2^{n-1}x)\| \le \varphi(2^{n-1}x, 2^{n-1}x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$. So

$$\begin{aligned}\|f(2^n\mu x) - \mu f(2^n x)\| \le& \|f(2^n\mu x) - 2\mu f(2^{n-1}x)\| + \|2\mu f(2^{n-1}x) - \mu f(2^n x)\|\\ \le& \varphi(2^{n-1}x, 2^{n-1}x) + \varphi(2^{n-1}x, 2^{n-1}x)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$. Thus $2^{-n}\|f(2^n\mu x) - \mu f(2^n x)\| \to 0$ as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$. Hence

$$\Theta(\mu x) = \lim_{n\to\infty} \frac{f(2^n\mu x)}{2^n} = \lim_{n\to\infty} \frac{\mu f(2^n x)}{2^n} = \mu\Theta(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [3, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. And $\Theta(x) = \Theta(3 \cdot \frac{1}{3}x) = 3\Theta(\frac{1}{3}x)$ for all $x \in \mathcal{B}$. So $\Theta(\frac{1}{3}x) = \frac{1}{3}\Theta(x)$ for all $x \in \mathcal{B}$. Thus

$$\begin{aligned}
\Theta(\lambda x) &= \Theta(\frac{M}{3} \cdot 3\frac{\lambda}{M}x) = M \cdot \Theta(\frac{1}{3} \cdot 3\frac{\lambda}{M}x) = \frac{M}{3}\Theta(3\frac{\lambda}{M}x) \\
&= \frac{M}{3}\Theta(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(\Theta(\mu_1 x) + \Theta(\mu_2 x) + \Theta(\mu_3 x)) \\
&= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)\Theta(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}\Theta(x) \\
&= \lambda\Theta(x)
\end{aligned}$$

for all $x \in \mathcal{B}$. Hence

$$\Theta(\zeta x + \eta y) = \Theta(\zeta x) + \Theta(\eta y) = \zeta\Theta(x) + \eta\Theta(y)$$

for all $\zeta, \eta \in \mathbb{C}(\zeta, \eta \neq 0)$ and all $x, y \in \mathcal{B}$. And $\Theta(0x) = 0 = 0\Theta(x)$ for all $x \in \mathcal{B}$. So the additive mapping $\Theta : \mathcal{B} \to \mathcal{C}$ is a $\mathbb{C}$-linear mapping.

Since $f(xy) = f(x)f(y)$ for all $x, y \in \mathcal{B}$,

$$\Theta(xy) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n xy) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)f(y) = \Theta(x)f(y) \tag{3}$$

for all $x, y \in \mathcal{B}$. By the additivity of Θ and (3),

$$2^n\Theta(xy) = \Theta(2^n xy) = \Theta(x(2^n y)) = \Theta(x)f(2^n y)$$

for all $x \in \mathcal{B}$. Hence

$$\Theta(xy) = \frac{1}{2^n}\Theta(x)f(2^n y) = \Theta(x)\frac{1}{2^n}f(2^n y) \tag{4}$$

for all $x, y \in \mathcal{B}$. Taking the limit in (4) as $n \to \infty$, we obtain

$$\Theta(xy) = \Theta(x)\Theta(y)$$

for all $x, y \in \mathcal{B}$. So the additive mapping $\Theta : \mathcal{B} \to \mathcal{C}$ is an algebra homomorphism. By (3),

$$\Theta(e)\Theta(x) = \Theta(ex) = \Theta(e)f(x)$$

for all $x \in \mathcal{B}$. Since $lim_{n\to\infty} \frac{f(2^n e)}{2^n} = \Theta(e)$ is invertible,

$$\Theta(x) = f(x)$$

for all $x \in \mathcal{B}$. Since the mapping $f : \mathcal{B} \to \mathcal{C}$ is bijective, the additive mapping $\Theta : \mathcal{B} \to \mathcal{C}$ is bijective. So the bijective mapping $f = \Theta : \mathcal{B} \to \mathcal{C}$ is an algebra isomorphism, as desired. □

Corollary 2. *Let $f : \mathcal{B} \to \mathcal{C}$ be a bijective mapping satisfying $f(xy) = f(x)f(y)$ and $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|f(\mu x + \mu y) - \mu f(x) - \mu f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in \mathcal{B}$. Assume that $\lim_{n\to\infty} \frac{f(2^n e)}{2^n}$ is invertible. Then the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra isomorphism.

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 1. □

Theorem 3. *Let $f : \mathcal{B} \to \mathcal{C}$ be a bijective mapping satisfying $f(xy) = f(x)f(y)$ and $f(0) = 0$ for which there exists a function $\varphi : \mathcal{B} \times \mathcal{B} \to [0, \infty)$ satisfying* (i) *and* (iii) *such that*

$$\text{(v)} \qquad \|f(\mu x + \mu y) - \mu f(x) - \mu f(y)\| \leq \varphi(x, y)$$

for $\mu = 1, i$, and all $x, y \in \mathcal{B}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{B}$, then the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra isomorphism.

Proof. Put $\mu = 1$ in (v). By the same reasoning as the proof of Theorem 1, there exists an additive mapping $\Theta : \mathcal{B} \to \mathcal{C}$ satisfying the inequality (iv). By the same reasoning as the proof of [6, Theorem], the additive mapping $\Theta : \mathcal{B} \to \mathcal{C}$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (v). By the same method as the proof of Theorem 1, one can obtain that

$$\Theta(ix) = \lim_{n\to\infty} \frac{f(2^n ix)}{2^n} = \lim_{n\to\infty} \frac{if(2^n x)}{2^n} = i\Theta(x)$$

for all $x \in \mathcal{B}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned}\Theta(\lambda x) &= \Theta(sx + itx) = s\Theta(x) + t\Theta(ix) = s\Theta(x) + it\Theta(x)\\ &= \lambda\Theta(x)\end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{B}$. So

$$\Theta(\zeta x + \eta y) = \Theta(\zeta x) + \Theta(\eta y) = \zeta\Theta(x) + \eta\Theta(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{B}$. Hence the additive mapping $\Theta : \mathcal{B} \to \mathcal{C}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as the proof of Theorem 1. □

3. Stability of $*$-isomorphisms between unital C^*-algebras

In this section, let $\mathcal{B}$ be a unital C^*-algebra with unitary group $\mathcal{U}(\mathcal{B})$, and $\mathcal{C}$ a unital C^*-algebra.

We are going to show the generalized Hyers-Ulam-Rassias stability of $*$-isomorphisms between unital C^*-algebras.

Theorem 4. *Let $f : \mathcal{B} \to \mathcal{C}$ be a bijective mapping satisfying $f(xy) = f(x)f(y)$ and $f(0) = 0$ for which there exists a function $\varphi : \mathcal{B} \times \mathcal{B} \to [0, \infty)$ satisfying* (i), (ii), *and* (iii) *such that*

(vi) $$\|f(2^n u^*) - f(2^n u)^*\| \le \varphi(2^n u, 2^n u),$$

for all $u \in \mathcal{U}(\mathcal{B})$ and all $n = 0, 1, \cdots$. Then the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra $*$-isomorphism.

Proof. By the same reasoning as the proof of Theorem 1, there exists a $\mathbb{C}$-linear mapping $\Theta : \mathcal{B} \to \mathcal{C}$ satisfying the inequality (iv), and the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra isomorphism with $f = \Theta$. The $\mathbb{C}$-linear mapping $\Theta : \mathcal{B} \to \mathcal{C}$ is given by

$$\Theta(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in \mathcal{B}$.

By (2) and (vi), we get

$$\begin{aligned}\Theta(u^*) &= \lim_{n\to\infty} \frac{f(2^n u^*)}{2^n} = \lim_{n\to\infty} \frac{f(2^n u)^*}{2^n} = (\lim_{n\to\infty} \frac{f(2^n u)}{2^n})^*\\ &= \Theta(u)^*\end{aligned}$$

for all $u \in \mathcal{U}(\mathcal{B})$. Since Θ is $\mathbb{C}$-linear and each $x \in \mathcal{B}$ is a finite linear combination of unitary elements (see [4, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^{m} \lambda_j u_j, \quad \lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{B})$,

$$\begin{aligned}\Theta(x^*) &= \Theta(\sum_{j=1}^{m} \overline{\lambda_j} u_j^*) = \sum_{j=1}^{m} \overline{\lambda_j}\Theta(u_j^*) = \sum_{j=1}^{m} \overline{\lambda_j}\Theta(u_j)^* = (\sum_{j=1}^{m} \lambda_j \Theta(u_j))^* \\ &= \Theta(\sum_{j=1}^{m} \lambda_j u_j)^* = \Theta(x)^*\end{aligned}$$

for all $x \in \mathcal{B}$. Hence the algebra isomorphism $\Theta : \mathcal{B} \to \mathcal{C}$ is a $*$-mapping. So the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra $*$-isomorphism, as desired. □

Corollary 5. *Let $f : \mathcal{B} \to \mathcal{C}$ be a bijective mapping satisfying $f(xy) = f(x)f(y)$ and $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}\|f(\mu x + \mu y) - \mu f(x) - \mu f(y)\| &\leq \theta(\|x\|^p + \|y\|^p), \\ \|f(2^n u^*) - f(2^n u)^*\| &\leq 2 \cdot 2^{np}\theta\end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{B})$, all $n = 0, 1, \cdots$, and all $x, y \in \mathcal{B}$. Assume that $\lim_{n\to\infty} \frac{f(2^n e)}{2^n}$ is invertible. Then the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra $*$-isomorphism.

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 4. □

Theorem 6. *Let $f : \mathcal{B} \to \mathcal{C}$ be a bijective mapping satisfying $f(xy) = f(x)f(y)$ and $f(0) = 0$ for which there exists a function $\varphi : \mathcal{B} \times \mathcal{B} \to [0, \infty)$ satisfying* (i), (iii), (v), *and* (vi). *If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{B}$, then the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra $*$-isomorphism.*

Proof. By the same reasoning as the proof of Theorem 1, there exists a $\mathbb{C}$-linear mapping $\Theta : \mathcal{B} \to \mathcal{C}$ satisfying the inequality (iv), and the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra isomorphism with $f = \Theta$.

By the same method as the proof of Theorem 4, we obtain that the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra $*$-isomorphism, as desired. □

4. Stability of $*$-isomorphisms between unital JB^*-algebras

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [7]). Let $\mathcal{H}$ be a complex Hilbert space, regarded as the "state space" of a quantum mechanical system. Let $\mathcal{L}(\mathcal{H})$ be the real vector space of all bounded self-adjoint linear operators on $\mathcal{H}$, interpreted as the (bounded) *observables* of the system. In 1932, Jordan observed that $\mathcal{L}(\mathcal{H})$ is a (nonassociative) algebra via the *anti-commutator product* $x \circ y := \frac{xy+yx}{2}$. A commutative algebra X with product $x \circ y$ is called a *Jordan algebra* if $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$ holds.

A complex Jordan algebra $\mathcal{B}$ with product $x \circ y$, unit element e and involution $x \mapsto x^*$ is called a JB^*-algebra if $\mathcal{B}$ carries a Banach space norm $\| \cdot \|$ satisfying $\|x \circ y\| \le \|x\| \cdot \|y\|$ and $\|\{xx^*x\}\| = \|x\|^3$. Here $\{xy^*z\} := x \circ (y^* \circ z) - y^* \circ (z \circ x) + z \circ (x \circ y^*)$ denotes the *Jordan triple product* of $x, y, z \in \mathcal{B}$. Throughout this section, let $\mathcal{B}$ and $\mathcal{C}$ be unital JB^*-algebras.

We are going to show the generalized Hyers-Ulam-Rassias stability of $*$-isomorphisms between unital JB^*-algebras.

Theorem 7. *Let $f : \mathcal{B} \to \mathcal{C}$ be a bijective mapping satisfying $f(x \circ y) = f(x) \circ f(y)$ and $f(0) = 0$ for which there exists a function $\varphi : \mathcal{B} \times \mathcal{B} \to [0, \infty)$ satisfying* (i) *and* (ii) *such that*

$$\|f(x^*) - f(x)^*\| \le \varphi(x, x) \tag{vii}$$

for all $x \in \mathcal{B}$. Assume that (viii) $\lim_{n\to\infty} \frac{f(2^n e)}{2^n} = e'$. *Then the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra $*$-isomorphism.*

Proof. By the same reasoning as the proof of Theorem 1, there exists a $\mathbb{C}$-linear mapping $\Theta : \mathcal{B} \to \mathcal{C}$ satisfying the inequality (iv), and the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra isomorphism with $f = \Theta$. The $\mathbb{C}$-linear mapping $\Theta : \mathcal{B} \to \mathcal{C}$ is given by

$$\Theta(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in \mathcal{B}$.

It follows from (2) and (vii) that

$$\begin{aligned}\Theta(x^*) &= \lim_{n\to\infty} \frac{f(2^n x^*)}{2^n} \\ &= \lim_{n\to\infty} \frac{f(2^n x)^*}{2^n} = (\lim_{n\to\infty} \frac{f(2^n x)}{2^n})^* \\ &= \Theta(x)^*\end{aligned}$$

for all $x \in \mathcal{B}$. Hence the algebra isomorphism $\Theta : \mathcal{B} \to \mathcal{C}$ is a $*$-mapping. So the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra $*$-isomorphism, as desired. $\square$

Corollary 8. *Let $f : \mathcal{B} \to \mathcal{C}$ be a bijective mapping satisfying $f(x \circ y) = f(x) \circ f(y)$ and $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|f(\mu x + \mu y) - \mu f(x) - \mu f(y)\| \leq \theta(\|x\|^p + \|y\|^p),$$
$$\|f(x^*) - f(x)^*\| \leq 2\theta\|x\|^p$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in \mathcal{B}$. Assume that

$$\lim_{n\to\infty} \frac{f(2^n e)}{2^n} = e'.$$

Then the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra $*$-isomorphism.

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 7. □

Theorem 9. *Let $f : \mathcal{B} \to \mathcal{C}$ be a bijective mapping satisfying $f(x \circ y) = f(x) \circ f(y)$ and $f(0) = 0$ for which there exists a function $\varphi : \mathcal{B} \times \mathcal{B} \to [0, \infty)$ satisfying* (i), (v), (vii), *and* (viii). *If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{B}$, then the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra $*$-isomorphism.*

Proof. By the same reasoning as the proof of Theorem 1, there exists a $\mathbb{C}$-linear mapping $\Theta : \mathcal{B} \to \mathcal{C}$ satisfying the inequality (iv), and the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra isomorphism with $f = \Theta$.

By the same method as the proof of Theorem 7, we obtain that the bijective mapping $f : \mathcal{B} \to \mathcal{C}$ is an algebra $*$-isomorphism, as desired. □

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