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LIPSCHITZ STABILITY OF IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS BY RAZUMIKHIN METHOD

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Abstract. This paper studies uniform Lipschitz stability for impulsive functional differential equations. Several criteria on uniform Lipschitz stability are established by using the method of Lyapunov and the Razumikhin technique. Some examples are also worked out to illustrate our results.

1. Introduction and preliminaries

The notion of Lipschitz stability was proposed by Dannan and Elaydi in [3], where some sufficient conditions for Lipschitz stability were given for ordinary differential equations and the relation between Lipschitz stability and other type of Lyapunov stability was investigated. It is shown that this notion lies between uniform stability and asymptotic stability in variation. But it neither implies asymptotic stability nor is implied by it. An important feature is that, unlike uniform stability, the linearized system preserves the property of Lipschitz stability from the original nonlinear system [3,4]. The Lipschitz stability criteria have been extended to integro-differential equations in [6] and functional differential equations in [5,7]. But to the best of our knowledge, Lipschitz stability results are not yet available for impulsive functional differential equations.

The objective of this paper is to study the problem of Lipschitz stability for impulsive functional differential equations, incorporating the ideas developed recently in [1,8,9,11,14,15]. Several criteria on uniform Lipschitz stability are established by using the method of Lyapunov and the Razumikhin technique. Moreover, it is shown that impulses play an important role in stabilization

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of unstable systems. We also establish a stability theorem by using multiple Lyapunov functions based on the idea developed in [16]. Some examples are also worked out to illustrate our results.

Consider the impulsive functional differential equation

$$\begin{cases} x' = f(t, x_t), \quad t \neq t_k, \\ x(t_k) = J_k(x(t_k^-)), \quad k \in N, \end{cases} \tag{1.1}$$

where N is the set of positive integers, $f : [t_0, \infty) \times PC \to R^n$ and $J_k(x) : R^n \to R^n$ for all $k \in N$ and $PC = PC([-\tau, 0], R^n) = \{\Phi : [-\tau, 0] \to R^n, \Phi(t)$ is continuous everywhere except at a finite number of points $\bar{t}$ at which $\Phi(\bar{t}^+)$ and $\Phi(\bar{t}^-)$ exist and $\Phi(\bar{t}^+) = \Phi(\bar{t})$ $\}$, $\tau > 0$ is the upper bound of time delays of our systems, $t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ with $t_k \to \infty$ as $k \to \infty$ and $x'(t)$ denotes the right hand derivative of $x(t)$.

For any $t \geq t_0, x_t \in PC$ is defined by $x_t(s) = x(t + s), -\tau \leq s \leq 0$. For $\phi \in PC$, the norm of ϕ is defined by $\| \phi \| = \sup_{-\tau \leq s \leq 0} | \phi(s) |$, where $|x| = \max_{1 \leq i \leq n}\{x_i\}$ for any $x = (x_1, x_2, \cdots, x_n) \in R^n$.

We assume that $f(t, 0) \equiv 0$ and $J_k(0) \equiv 0$, so that equation (1.1) admits the zero solution.

For any $t_0 \in R^+$ and $\phi \in PC$, the initial value problem of equation (1.1) is given by

$$\begin{cases} x'(t) = f(t, x_t), \quad t \neq t_k, \\ x(t_k) = J_k(x(t_k^-)), \quad k \in N, \\ x_{t_0} = \phi. \end{cases} \tag{1.2}$$

A function $x(t) : [t_0 - \tau, \infty) \to R^n$ with $x_{t_0} = \phi$ is said to be a solution of system (1.2), if it is continuous and satisfies the differential equation $x'(t) = f(t, x_t)$ in each $[t_i, t_{i+1}), i = 0, 1, \cdots$, and at $t = t_i$ it satisfies $x(t_i) = J_i\left(x\left(t_i^-\right)\right)$.

We shall make the following assumptions.

(H1) $f(t, \psi)$ is *composite-PC*, i.e., if for each $t_0 \in R_+$ and $\alpha > 0$, where $[t_0, t_0 + \alpha] \in R_+$, if $x \in PC([t_0 - \tau, t_0 + \alpha], R^n)$ and x is continuous at each $t \neq t_k$ in $(t_0, t_0 + \alpha]$, then the composite function g defined by $g(t) = f(t, x_t)$ is an element of the function class $PC([t_0, t_0 + \alpha], R^n)$.

(H2) $f(t, \psi)$ is *quasi-bounded*, i.e., if for each $t_0 \in R_+$ and $\alpha > 0$, where $[t_0, t_0 + \alpha] \in R_+$, and for each compact set $F \in R^n$ there exists some $M > 0$ such that $\|f(t, \psi)\| \leq M$ for all $(t, \psi) \in [t_0, t_0 + \alpha] \times PC([-\tau, 0], F)$.

(H3) For each fixed $t \in R_+$, $f(t, \psi)$ is continuous on $PC([-\tau, 0], R^n)$.

It is shown in [2] that under Assumptions $(H1)$-$(H3)$, the initial value problem (1.1) has a solution $x(t, t_0, \phi) \triangleq x(t)$ existing in a maximal interval I. If, in addition, $f(t, \psi)$ is locally Lipschitz in ψ, then the solution is unique.

Definition 1.1. The function $V(t, x) : [t_0 - \tau, \infty) \times R^n \to R^+$ belongs to class ν_0^n if

(A1) V is continuous on each of the sets $[t_{k-1}, t_k) \times R^n$ and for all $x,\ y \in R^n$ and $k \in N$, $\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$ exists.

(A2) $V(t, x)$ is locally Lipschitzian in $x \in R^n$, and for all $t \geq t_0$, $V(t, 0) \equiv 0$.

Definition 1.2. Given a function $V :\ [t_0 - \tau, \infty) \times R^n \to R^+$, the upper right-hand derivative of V with respect to system (1.1) is defined by

$$D^+V(t, x) = \limsup_{\alpha\to 0^+} \frac{1}{\alpha}[V(t + \alpha, x + \alpha f(t, x)) - V(t, x)],$$

for $(t, x) \in [t_0 - \tau, \infty) \times R^n$.

Definition 1.3. The zero solution of (1.1) is said to be uniformly Lipschitz stable through $(t_0, \phi) \in R^+ \times PC$, if there exists a constant $\eta > 0$ independent of t_0 and $M = M(\eta) \geq 1$, such that

$$\mid x(t, t_0, \phi) \mid \leq M \cdot \parallel \phi \parallel, \quad \text{for } t \geq t_0 \text{ and } \parallel \phi \parallel < \eta.$$

Remark 1.1. From Definition 1.3, we know that uniformly Lipschitz stability implies uniform stability.

We define the following sets for later use.

$$\begin{aligned}
S(\rho) &= \{x \in R^n : \mid x \mid < \rho,\ \text{for } \rho > 0\},\\
K_0 &= \left\{H \in C(R^+, R^+) :\ H(0) = 0,\ H(s) > 0 \text{ for } s > 0\right\},\\
K &= \left\{\omega \in C(R^+, R^+) : \text{strictly increasing and } \omega(0) = 0\right\},\\
K_1 &= \{\psi \in K :\ \psi(s) < s \text{ for s} > 0\},\\
K_2 &= \{\phi \in K :\ \phi(u) \geq u \text{ for u} > 0\},\\
\Omega &= \Big\{\omega(t, u) :\ \omega \in C([t_{k-1}, t_k) \times R^+, R^+),\ k \in N;\ \text{for each } x \in R^+\\
&\qquad \text{and } k \in N, \lim_{(t,u)\to(t_k^-,x)} \omega(t, u) = \omega(t_k^-, x) \text{ exists}\Big\}.
\end{aligned}$$

2. Stability criteria

We shall establish some Lipschitz stability criteria in this section by the method of Lyapunov function and Razumkhin technique. Our first result utilizes the comparison principle.

Theorem 2.1. *Assume that there exist functions* $V \in \nu_0^n,\ \omega_1 \in K,\ g \in \Omega$ *and* $\psi_k \in K_2$ *such that*

(i) $V(t_k, J_k(x)) \leq \psi_k(V(t_k^-, x(t_k^-))),\ k \in N$;

(ii) $\omega_1(\mid x \mid) \leq V(t,x)$, *for any* $\|\phi\| < \eta$, *there exist constant* $L = L(\eta) > 0$ *and function* q *with* $q(L) \geq 1$ *for any* $L \geq 1$ *such that* $\omega_1^{-1}(L \mid x \mid) \leq q(L) \mid x \mid$, *where* ω_1^{-1} *is the inverse function of* ω_1;

(iii) *for any solution* $x(t)$ *of (1.1),* $V(t+s, x(t+s)) \leq V(t, x(t)),\ -\tau \leq s \leq 0,\ (t,x) \in [t_{k-1}, t_k) \times S(\rho)$ *implies that*

$$D^+V(t,x(t)) \leq g(t, V(t,x(t)));$$

(iv) *the zero solution of the impulsive scalar equation*

$$\begin{cases} u' = & g(t,u), \quad t \geq t_0, \\ u(t_k) = & \psi_k(u(t_k^-)), \quad k \in N, \\ u(t_0) = & u_0 \geq 0, \end{cases} \tag{2.1}$$

is uniformly Lipschitz stable, where u_0 *is a constant such that* $u_0 = \max_{t_0-\tau\leq s\leq t_0}\{V(s)\}$.

Then the zero solution of (1.1) is uniformly Lipschitz stable.

Proof. By condition (i) and (iii), it follows by Lemma 3.1 of [12] that

$$V(t,x) \leq u(t, t_0, u_0), \tag{2.2}$$

where $u(t,t_0,u_0)$ is the maximal solution of (2.1).

Since the zero solution of (2.1) is uniformly Lipschitz stable, then there exists $\eta > 0,\ M = M(\eta) > 0$ such that

$$u_1(t, t_0, u_0) \leq M \cdot u_0, \tag{2.3}$$

where $u_1(t,t_0,u_0)$ is the solution of (2.1) with $u_1(t_0,t_0,u_0) = u_0$.

Choose $M(\eta) \geq 1$ such that $\|\phi\| < \eta$ implies

$$u_0 \leq M\cdot \parallel \phi \parallel \tag{2.4}$$

By the inequalities (2.2)-(2.4) we get

$$\omega_1(\mid x \mid) \leq V(t,x) \leq u(t,t_0,u_0) \leq M \cdot u_0 \leq M^2 \cdot \parallel \phi \parallel .$$

By condition (ii) we have, for any solution $x(t)$ of (1.1) with $\|\phi\| < \eta$

$$\mid x \mid \leq \omega_1^{-1}(M^2 \cdot \parallel \phi \parallel) \leq q(M^2) \cdot \parallel \phi \parallel, \ q(M^2) \geq 1,$$

which completes the proof. □

An interesting special case is given in the following corollary.

Corollary 2.1. *Assume that there exist functions $V \in \nu_0^n, \ \omega_1 \in K, \ g \in \Omega$ such that*

(i) $V(t_k, J_k(x)) \leq (1+b_k)(V(t_k^-, x(t_k^-))), \ k \in N,$ *where* $b_k > 0$ *and* $\sum_{k=1}^{\infty} b_k < \infty$*;*

(ii) $\omega_1(\mid x \mid) \leq V(t,x)$*, for any* $\|\phi\| < \eta$*, there exist constant* $L = L(\eta) > 0$ *and function* q *with* $q(L) \geq 1$ *for any* $L \geq 1$ *such that* $\omega_1^{-1}(L \mid x \mid) \leq q(L) \mid x \mid$*, where* ω_1^{-1} *is the inverse function of* ω_1*;*

(iii) *for any solution* $x(t)$ *of (1.1),* $V(t+s, x(t+s)) \leq V(t, x(t)), \ -\tau \leq s \leq 0, \ (t,x) \in [t_{k-1}, t_k) \times S(\rho)$ *implies that*

$$D^+V(t,x(t)) \leq 0;$$

Then the zero solution of (1.1) with the initial function ϕ *satisfying* $\|\phi\| \geq \max_{t_0-\tau \leq s \leq t_0}\{V(s,\phi(s))\}$ *is uniformly Lipschitz stable.*

Proof. Choose $\psi_k(s) = (1+b_k)s$ for any $s \in R^+, \ k \in N$ and $g(t,u) \equiv 0$ for any $t, u \in R^+$ in Theorem 2.1. □

Our next result incorporates the positive effects of the impulses.

Theorem 2.2. *Assume that there exist functions $V \in \nu_0^n, \ \omega_1, \ \omega_2 \in K, \ H \in K_0$ and $\psi \in K_1$ such that*

(i) $\omega_1(\mid x \mid) \leq V(t,x) \leq \omega_2(\mid x \mid)$*, and there exist* $\eta > 0$ *and* $M = M(\eta) \geq 1$ *such that* $\|\phi\| < \eta$ *implies that*

$$\psi^{-1}(\omega_2(\|\phi\|)) \leq \omega_1(M \cdot \|\phi\|),$$

where ψ^{-1} *is the inverse function of* ψ*;*

(ii) $V(t_k, J_k(x(t_k^-))) \leq \psi(V(t_k^-, x(t_k^-))), \ k \in N;$

(iii) *for any solution* $x(t)$ *of (1.1),* $V(t+s, x(t+s)) \le \psi^{-1}(V(t, x(t))), -\tau \le s \le 0,\ (t, x) \in [t_{k-1}, t_k) \times S(\rho)$ *implies that*

$$D^+V(t, x(t)) \le g(t)H(V(t, x(t))),$$

where $g : [t_0, \infty) \to R^+$ *is locally integrable;*

(iv) H *is nondecreasing and for all* $k \in N$ *and any* $\mu > 0$*,*

$$\int_{\mu}^{\psi^{-1}(\mu)} \frac{du}{H(u)} > \int_{t_{k-1}}^{t_k} g(s)ds.$$

Then the zero solution of (1.1) is uniformly Lipschitz stable.

Proof. Let $V(t) = V(t, x(t))$, where $x(t) = x(t, t_0, \phi)$ is the solution of (1.1) through (t_0, ϕ) with $\|\phi\| < \eta,\ t_0 \in R^+$. Then

$$\begin{aligned}\omega_1(|\ x(t)\ |) \le V(t, x) \le \omega_2(|\ x(t)\ |) \le \omega_2(\|\phi\|) \\ \le \psi^{-1}(\omega_2(\|\phi\|)),\quad t_0 - \tau \le t \le t_0.\end{aligned}$$

We claim that

$$V(t) \le \psi^{-1}(\omega_2(\|\phi\|)), \quad t_0 \le t < t_1. \tag{2.5}$$

Otherwise, there exists a $\bar{t} \in (t_0, t_1)$ such that

$$V(\bar{t}) > \psi^{-1}(\omega_2(\|\phi\|)) > \omega_2(\|\phi\|) \ge V(t_0),$$

which implies that there is a $t^* \in (t_0, \bar{t})$ such that

$$V(t^*) = \psi^{-1}(\omega_2(\|\phi\|)), \quad V(t) \le \psi^{-1}(\omega_2(\|\phi\|)), \quad t_0 - \tau \le t \le t^*,$$

and there exists a $\underline{t} \in [t_0, t^*)$ such that

$$V(\underline{t}) = \omega_2(\|\phi\|), \quad V(t) \ge \omega_2(\|\phi\|), \quad \underline{t} \le t \le t^*,$$

therefore, for all $t \in [\underline{t}, t^*]$,

$$V(t+s) \le \psi^{-1}(\omega_2(\|\phi\|)) \le \psi^{-1}(V(t)),\ -\tau \le s \le 0,$$

choose $\rho > 0$ such that $\psi^{-1}(\omega_2(\eta)) < \omega_1(\rho)$, then we have, for all $t \in [\underline{t}, t^*]$

$$\omega_1(|x(t)|) \le V(t) \le \psi^{-1}(\omega_2(\|\phi\|)) \le \psi^{-1}(\omega_2(\eta)),$$

i.e., $|x(t)| < \rho$ for all $t \in [\underline{t}, t^*]$. Thus by condition (iii) we have $V'(t, x(t)) \leq g(t)H(V(t, x(t)))$, $\underline{t} \leq t \leq t^*$, so

$$\int_{V(\underline{t})}^{V(t^*)} \frac{du}{H(u)} \leq \int_{\underline{t}}^{t^*} g(s)ds \leq \int_{t_0}^{t_1} g(s)ds,$$

i.e.

$$\int_{\omega_2(\|\phi\|)}^{\psi^{-1}(\omega_2(\|\phi\|))} \frac{du}{H(u)} \leq \int_{t_0}^{t_1} g(s)ds,$$

a contradiction and hence (2.5) holds. From (2.5) and condition (ii), we get

$$V(t_1) = V(t_1, J_1(x(t_1^-))) \leq \psi(V(t_1^-))) \leq \omega_2(\|\phi\|), \tag{2.6}$$

In a similar way as in the proof of (2.5) and (2.6) we can get

$$V(t) \leq \psi^{-1}(\omega_2(\|\phi\|)), \quad t_1 \leq t < t_2, \quad V(t_2) \leq \omega_2(\|\phi\|).$$

By a simple induction and the fact $s < \psi^{-1}(s)$, we can prove in general that

$$V(t) \leq \psi^{-1}(\omega_2(\|\phi\|)), \quad t_m \leq t \leq t_{m+1}, \quad m = 0, 1, 2 \cdots,$$

which, together with (2.5), yields

$$\omega_1(|\ x\ |) \leq V(t) \leq \psi^{-1}(\omega_2(\|\phi\|)) \leq \omega_1(M \cdot \|\phi\|), \quad t \geq t_0,$$

which completes the proof. □

Remark 2.1. It should be noted that the underlying system without impulses may be unstable. Theorem 2.2 shows that impulses can be used to stabilize an unstable system.

3. Method of multiple Lyapunov functions

In this section, we shall establish a Razumkhin-type theorem with multiple Lyapunov functions.

In what follows, we separate $x = (x_1, x_2, \cdots, x_n)^T$ into several vectors, i.e.

$$x = (x^{(1)}, x^{(2)}, \cdots, x^{(m)})^T,$$

where $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, \cdots, x_{n_j}^{(j)})$, $j = 1, 2, \cdots, m$, and $\sum_{j=1}^m n_j = n$. We denote the norms by $|x^{(j)}| = \max_{1 \leq k \leq n_j} \left\{ |x_k^{(j)}| \right\}$, $j = 1, 2, \cdots, m$, and $|x| = \max_{1 \leq j \leq m} \left\{ |x^{(j)}| \right\}$.

Theorem 3.1. *Assume that there exist functions $V_j(t, x^{(j)}) \in \nu_0^{n_j}$, a_j, $b_j \in K$, $H_j \in K_0$, $j = 1, 2, \cdots, m$, such that*

(i) $a_j(|x^{(j)}|) \le V_j(t, x^{(j)}) \le b_j(|x^{(j)}|)$, *and for any* $\eta > 0$ *there exist constant* $L = L(\eta) > 0$ *and function* q *with* $q(L) \ge 1$ *for any* $L \ge 1$ *such that* $\|\phi\| < \eta$ *implies* $\max_{1\le i\le m}\{a_i^{-1}(L\sum_{k=1}^m b_k(\|\phi\|))\} \le q(L)\|\phi\|$;

(ii) *for all* $k \in N$, $V_j(t_k^-, x^{(j)}(t_k^-)) = \max_{1\le i\le m}\{V_i(t_k^-, x^{(i)}(t_k^-))\}$ *implies*

$$\max_{1\le i\le m}\{V_i(t_k, x^{(i)}(t_k))\} \le (1 + d_k)V_j(t_k^-, x^{(j)}(t_k^-)),$$

where $d_k \ge 0$ *with* $\sum_{i=1}^{\infty} d_k < \infty$;

(iii) *for any solution* $x(t) = (x^{(1)}(t), x^{(2)}(t), \cdots, x^{(m)}(t))$ *of (1.1), and for any* β_j $\gamma_j > 0$, *there exist* $\lambda_j = \lambda_j(\beta_j, \gamma_j) > 0$, *such that* $V_j(t + s, x^{(j)}(t + s)) \le V_j(t, x^{(j)}(t))$ *for* $s \in [-\tau, 0]$, $t \in [t_{k-1}, t_k)$ *and* $\gamma_j \le |x^{(j)}(t)| \le \beta_j$ *imply*

$$D^+V_j(t, x^{(j)}(t)) \le -H_j(|x^{(j)}(t)|) + \lambda_j,$$

where $V_j(t, x^{(j)}(t)) = \max_{1\le i\le m}\{V_i(t, x^{(i)}(t))\}$.

Then the zero solution of (1.1) is uniformly Lipschitz stable.

Proof. Let $x(t) = x(t, t_0, \phi)$ be any solution of (1.1) with $\|\phi\| < \eta$ for some $\eta > 0$, define $V(t) = \max_{1\le i\le m}\{V_i(t)\}$, where $V_i(t) = V_i(t, x^{(i)}(t))$ for $i = 1, 2, \cdots, m$.

We claim that

(C1). $\frac{1}{m}\sum_{i=1}^m a_i(|x^{(i)}(t)|) \le V(t) \le \sum_{i=1}^m b_i(|x^{(i)}(t)|)$;
(C2). $V(t_k) \le (1 + d_k)V(t_k^-)$, $k \in N$;
(C3). $V(t + s) \le V(t)$ for $s \in [-\tau, 0]$, $t \in [t_{k-1}, t_k)$ implies

$$D^+V(t) \le -H_j(|x^{(j)}(t)|) + \lambda_j,$$

where $j \in \left\{k \in \{1, 2, \cdots, m\} : V_k(t) = \max_{1\le i\le m}\{V_i(t)\}\right\}$.

Proof of $(C1)$ and $(C2)$:
By condition (i) and the definition of $V(t)$, we have

$$V(t) = \max_{1\le i\le m}\{V_i(t)\} \ge \frac{1}{m}\sum_{i=1}^m V_i(t) \ge \frac{1}{m}\sum_{i=1}^m a_i(|x^{(i)}(t)|),$$

and

$$V(t) = \max_{1 \le i \le m} \{V_i(t)\} \le \sum_{i=1}^{m} V_i(t) \le \sum_{i=1}^{m} b_i(|x^{(i)}(t)|),$$

which implies $(C1)$ holds. $(C2)$ comes directly from condition (ii) and the definition of $V(t)$.

Proof of $(C3)$:

Assume $V(t) = V_j(t)$ for $t \in [\alpha_{l_j k}, \beta_{l_j k}) \subset [t_{k-1}, t_k)$, where $[t_{k-1}, t_k) = \cup_{1 \le j \le m} [\alpha_{l_j k}, \beta_{l_j k})$. When $V(t+s) \le V(t)$ for $s \in [-\tau, 0]$, we have $V_j(t+s) \le V(t+s) \le V(t) = V_j(t)$, then by condition (iii) and the right continuity of $V(t)$, we obtain

$$\begin{aligned} D^+V(t) &= \limsup_{h \to 0^+} \frac{V(t+h) - V(t)}{h} = \limsup_{h \to 0^+} \frac{V_j(t+h) - V_j(t)}{h} \\ &= D^+V_j(t) \le -H_j(|x^{(j)}(t)|) + \lambda_j, \end{aligned}$$

which implies $(C3)$ holds.

Next, we shall prove the uniform Lipschitz stability of the zero solution of (1.1).

We claim that

$$V(t) \le \prod_{i=0}^{k-1} (1 + d_i) \sum_{i=1}^{m} b_i(\|\phi\|), \quad t \in [t_{k-1}, t_k), \ k \in N, \tag{3.1}$$

where $d_0 = 0$.

Firstly, for $t \in [t_0 - \tau, t_0]$, we have, by $(C1)$ and the fact $|x^{(i)}(t)| \le |x(t)| \le \|\phi\|$

$$V(t) \le \sum_{i=1}^{m} b_i(|x^{(i)}(t)|) \le \sum_{i=1}^{m} b_i(\|\phi\|). \tag{3.2}$$

Secondly, we show (3.1) holds for $k = 1$, i.e.

$$V(t) \le \sum_{i=1}^{m} b_i(\|\phi\|), \qquad t \in [t_0, t_1). \tag{3.3}$$

Suppose (3.3) is not true, then there exists some $\hat{t} \in [t_0, t_1)$ such that

$$V(\hat{t}) = \sum_{i=1}^{m} b_i(\|\phi\|), \quad V(\hat{t} + s) \le V(\hat{t}), \text{ for } s \in [-\tau, 0] \quad \text{and} \quad D^+V(\hat{t}) \ge 0, \tag{3.4}$$

since $V(t)$ is continuous in $t \in [t_0, t_1)$. Assume $V(\hat{t}) = V_j(\hat{t})$, by condition (i), $b_j(|x^{(j)}(\hat{t})|) \geq V(\hat{t}) = \sum_{i=1}^m b_i(\|\phi\|) \geq a_j(|x^{(j)}(\hat{t})|)$, i.e., $|x^{(j)}(\hat{t})| \geq b_j^{-1}(\sum_{i=1}^m b_i(\|\phi\|)) \triangleq \gamma_j$ and $|x^{(j)}(\hat{t})| \leq a_j^{-1}(\sum_{i=1}^m b_i(\|\phi\|)) \triangleq \beta_j$. Choose $0 < \lambda_j < \inf_{\gamma_j \leq s \leq \beta_j}\{H_j(s)\}$, then by $(C3)$ and (3.4) we have

$$\begin{aligned} D^+V(\hat{t}) &\leq -H_j(|x^j(t)|) + \lambda_j \\ &< \lambda_j + \lambda_j \ = \ 0, \end{aligned}$$

this contradicts (3.4), and hence (3.1) holds for $k = 1$.

Assume (3.1) holds for $k = p$, i.e.

$$V(t) \leq \prod_{i=0}^{p-1}(1 + d_i)\sum_{i=1}^m b_i(\|\phi\|), \qquad t \in [t_{p-1}, t_p), \tag{3.5}$$

then we have

$$V(t_p) \leq (1 + d_p)V(t_p^-) \leq \prod_{i=0}^{p}(1 + d_i)\sum_{i=1}^m b_i(\|\phi\|), \tag{3.6}$$

we now prove (3.1) holds for $k = p + 1$. Suppose not, then repeat the same argument as we prove (3.3), we will get a contradiction which shows (3.1) holds for $k = p + 1$. Then by induction, we know (3.1) is true.

Then we have, from (3.1) and condition (i),

$$V(t) \leq M\sum_{i=1}^m b_i(\|\phi\|),$$

where $M = \prod_{i=1}^\infty(1 + d_k)$, and then by $(C1)$

$$\begin{aligned} \frac{1}{m}a_i(|x^{(i)}(t)|) \leq \ & \frac{1}{m}\sum_{i=1}^m a_i(|x^{(i)}(t)|) \\ & \leq \ V(t) \ \leq \ M\sum_{i=1}^m b_i(\|\phi\|), \end{aligned}$$

that is.,

$$|x^{(i)}(t)| \leq a_i^{-1}(mM\sum_{i=1}^m b_i(\|\phi\|)),$$

so we have, for any solution $x(t)$ of (1.1) with $\|\phi\| < \eta$,

$$\begin{aligned}|x(t)| = & \max_{1\le i\le m}\{|x^{(i)}(t)|\}\\ \le & \max_{1\le i\le m}\{a_i^{-1}(mM\sum_{i=1}^{m} b_i(\|\phi\|))\}\\ \le & \ q(mM)\|\phi\|,\end{aligned}$$

where $q(mM) \ge 1$, this completes the proof. □

4. Examples

In this section, we shall discuss some examples to illustrate our results given in previous sections.

Example 4.1. Consider the impulsive nonlinear delay differential equation

$$\begin{aligned}x'(t) &= -2x(t-1)[1+x(t)], \quad t \ne \frac{k}{8}, \ k \in N,\\ x(t_k) &= \frac{1}{2}x(t_k^-), \quad t_k = \frac{k}{8}.\end{aligned} \tag{4.1}$$

Choose $V(t,x) = |x|, \ \psi(s) = \frac{1}{2}s, \ M = 2$, then condition (i) and (ii) of Theorem 2.2 hold.

Let $g(s) = 5s, \ H(s) = s, \ \rho = \frac{1}{4}$, then for any solution x(t) of (4.1), when $V(t+s, x(t+s)) \le \psi^{-1}(V(t,x(t)))$, i.e., $|x(t+s)| \le 2|x(t)|, \ s \in [-1,0]$, and $|x| < \rho$

$$\begin{aligned}D^+V(t,x(t)) &\le sgn(x(t))\{-2x(t-1)[1+x(t)]\}\\ &\le 2|x(t-1)|(1+|x(t)|)\\ &\le 5|x(t)| = 5V(t,x)\\ &\le g(t)V(t,x),\end{aligned}$$

and for any $\mu > 0$ and $k \in N$

$$\begin{aligned}\int_{\mu}^{\psi^{-1}(\mu)} \frac{du}{H(u)} &= \int_{\mu}^{2\mu} \frac{ds}{s} = ln2\\ &> \int_{t_{k-1}}^{t_k} g(s)ds = \frac{5}{8},\end{aligned}$$

which implies that condition (iii) and (iv) of Theorem 2.2 hold, then by Theorem 2.2, the zero solution of (4.1) is uniformly Lipschitz stable.

Remark 4.1. It was proven by Wright in 1995 that the zero solution of $x'(t) = -2x(t-1)[1+x(t)]$ is unstable(see [10,13]). Example 4.1 shows that appropriate impulses can make unstable system stable.

Example 4.2. Consider the impulsive nonlinear delay differential equation

$$\begin{aligned}
&x_1'(t) = -x_1(t) + e^{-t}x_3^{\frac{1}{3}}(t-2), \;\; t \neq k,\\
&x_2'(t) = cos(2t)x_1^{\frac{2}{3}}(t)x_2^{\frac{1}{3}}(t-1) - 2x_2(t), \;\; t \neq k,\\
&x_3'(t) = 3x_2(t-2) + 2x_3(t), \;\; t \neq k,\\
&x_1(k) = \frac{1}{3}x_1(k^-) - \frac{1}{3^k}x_2(k^-) + \frac{1}{5}x_3(k^-),\\
&x_2(k) = \frac{3}{2^k}x_2(k^-) - x_3(k^-),\\
&x_3(k) = -\frac{1}{4}x_1(k^-) + \frac{1}{2^k}x_2(k^-) - \frac{2}{3}x_3(k^-), \; k \in N.
\end{aligned} \tag{4.2}$$

We separate $x = (x_1, x_2, x_3)^T$ into two groups, i.e. $x = (x^{(1)}, x^{(2)})^T$, where $x^{(1)} = (x_1, x_3)$ and $x^{(2)} = x_2$, and choose $V_1(t, x^{(1)}) = |x^{(1)}| = \max\{x_1, x_3\}$ and $V_2(t, x^{(2)}) = |x^{(2)}| = |x_2|$.

Let $a_i(s) = b_i(s) = s$, $i = 1, 2$, then $a_i(|x^{(i)}|) \leq V_i(t, x^{(i)}) \leq b_i(|x^{(i)}|)$, and let $q(L) = 2L$, we have $a_i^{-1}(L\sum_{k=1}^{2} b_k(\|\phi\|)) = L(\|\phi\| + \|\phi\|) \leq q(L)\|\phi\|$, $i = 1, 2$, i.e., condition (i) of Theorem 3.1 is satisfied.

For any $k \in N$, if $V_1(t_k^-, x^{(1)}(t_k^-)) \geq V_2(t_k^-, x^{(2)}(t_k^-))$, that is, $\max\{|x_1(k^-)|, |x_3(\ k^-)|\} \geq |x_2(k^-)|$, then we have

$$\begin{aligned}
V_1(k, x^{(1)}(k)) &= \max\{|\frac{1}{3}x_1(k^-) - \frac{1}{3^k}x_2(k^-) + \frac{1}{5}x_3(k^-)|,\\
&\qquad |-\frac{1}{4}x_1(k^-) + \frac{1}{2^k}x_2(k^-) - \frac{2}{3}x_3(k^-)|\}\\
&\leq \frac{1}{3}|x_1(k^-)| + \frac{1}{2^k}|x_2(k^-)| + \frac{2}{3}|x_3(k^-)|\\
&\leq (1 + \frac{1}{2^k})\max\{|x_1(k^-)|, |x_3(k^-)|\}\\
&\leq (1 + \frac{1}{2^k})V_1(k^-, x^{(1)}(k^-)),
\end{aligned}$$

and

$$\begin{aligned}V_2(k,x^{(2)}(k)) &= |\frac{3}{2^k}x_2(k^-)-x_3(k^-)|\\ &\le \frac{3}{2^k}|x_2(k^-)|+|x_3(k^-)|\\ &\le (1+\frac{3}{2^k})\max\{|x_1(k^-)|,|x_3(k^-)|\}\\ &\le (1+\frac{3}{2^k})V_1(k^-,x^{(1)}(k^-)),\end{aligned}$$

which gives $\max_{1\le i\le 2}\{V_i(t_k,x^{(i)}(t_k))\}\le (1+\frac{3}{2^k})V_1(t_k^-,x^{(1)}(t_k^-))$.

On the other hand, if $V_2(t_k^-,x^{(2)}(t_k^-))\ge V_1(t_k^-,x^{(1)}(t_k^-))$, i.e., $|x_2(k^-)|\ge \max\{|x_1(k^-)|,\,|x_3(k^-)|\}$, then we have

$$\begin{aligned}V_1(k,x^{(1)}(k)) &= \max\{|\frac{1}{3}x_1(k^-)-\frac{1}{3^k}x_2(k^-)+\frac{1}{5}x_3(k^-)|,\\ &\qquad |-\frac{1}{4}x_1(k^-)+\frac{1}{2^k}x_2(k^-)-\frac{2}{3}x_3(k^-)|\}\\ &\le \frac{1}{3}|x_1(k^-)|+\frac{1}{2^k}|x_2(k^-)|+\frac{2}{3}|x_3(k^-)|\\ &\le (1+\frac{1}{2^k})|x_2(k^-)|\\ &\le (1+\frac{1}{2^k})V_2(k^-,x^{(2)}(k^-)),\end{aligned}$$

and

$$\begin{aligned}V_2(k,x^{(2)}(k)) &= |\frac{3}{2^k}x_2(k^-)-x_3(k^-)|\\ &\le \frac{3}{2^k}|x_2(k^-)|+|x_3(k^-)|\\ &\le (1+\frac{3}{2^k})|x_2(k^-)|\\ &\le (1+\frac{3}{2^k})V_2(k^-,x^{(2)}(k^-)),\end{aligned}$$

so we have

$$\max_{1\le i\le 2}\{V_i(t_k,x^{(i)}(t_k))\}\le (1+\frac{3}{2^k})V_2(t_k^-,x^{(2)}(t_k^-)).$$

Thus by choosing $d_k = \frac{3}{2^k}$, the condition (ii) of Theorem 3.1 is satisfied.

For any $\beta_1, \gamma_1 > 0$, choose $\lambda_1 = \max\{\beta_1^{\frac{1}{3}}, 5\beta_1\} > 0$, if $V_1(t, x^{(1)}(t)) \geq V_2(t, x^{(2)}(t))$, that is, $\max\{|x_1(t)|, |x_3(t)|\} \geq |x_2(t)|$, $V_1(t+s, x^{(1)}(t+s)) \leq V_1(t, x^{(1)}\ (t))$, that is, $\max\{|x_1(t+s)|, |x_3(t+s)|\} \leq \max\{|x_1(t), x_3(t)|\}$ for $s \in [-2, 0]$, and $\beta_1 \geq |x^{(1)}(t)| \geq \gamma_1$, then

1. for those t such that $|x^{(1)}(t)| = |x_1(t)|$, we have

$$\begin{aligned} D^+V_1(t, x^{(1)}(t)) &\leq sgn(x_1(t))x_1'(t) \\ &\leq -|x_1(t)| + |x_3^{\frac{1}{3}}(t-2)| \\ &\leq -|x_1(t)| + \beta_1^{\frac{1}{3}} \leq -|x^{(1)}(t)| + \beta_1^{\frac{1}{3}} \\ &\leq \lambda_1, \end{aligned}$$

2. for those t such that $|x^{(1)}(t)| = |x_3(t)|$, we have

$$\begin{aligned} D^+V_1(t, x^{(1)}(t)) &\leq sgn(x_3(t))x_3'(t) \\ &\leq 3|x_2(t-2)| + 2|x_3(t)| \\ &\leq 5|x^{(1)}(t)| \leq 5\beta_1 \leq \lambda_1, \end{aligned}$$

by choosing $H_1(s) \equiv 0$, we know condition (iii) of Theorem 3.1 holds for the case $V_1(t, x^{(1)}(t)) = \max\{V_1(t, x^{(1)}(t)), V_2(t, x^{(2)}(t))\}$.

By choosing $H_2(s) = s$ and $\lambda_2 = 1$, we have, if

$$V_2(t, x^{(2)}(t)) = \max\{V_1(t, x^{(1)}(t), V_2(t, x^{(2)}(t)\},$$

that is,

$$|x_2(t)| \geq \max\{|x_1(t)|, |x_3(t)|\}, V_2(t+s, x^{(2)}(t+s)) \leq V_2(t, x^{(2)}(t)),$$

that is,

$$|x_2(t+s)| \geq |x_2(t)| \quad \text{for} \quad s \in [-2, 0],$$

then

$$\begin{aligned} D^+V_2(t, x^{(2)}(t)) &\leq sgn(x_2(t))x_2'(t) \\ &\leq |x_1^{\frac{2}{3}}(t)| \times |x_2^{\frac{1}{3}}(t-1)| - 2|x_2(t)| \\ &\leq -|x_2(t)| \leq -H_2(|x^{(2)}(t)|) + \lambda_2, \end{aligned}$$

which implies condition (*iii*) of Theorem 3.1 holds, then by Theorem 3.1, the zero solution of (4.2) is uniformly Lipschitz stable.

Remark 4.2. Example 4.2 illustrates the advantages of using multiple Lyapunov functions which share the conditions usually imposed on a single Lyapunov function. Indeed, it would be difficult to find a single Lyapunov function for system (4.2) to satisfy all the necessary requirements.

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