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ITERATIVE PROCESS FOR CERTAIN NONLINEAR MAPPINGS IN UNIFORMLY SMOOTH BANACH SPACES

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Abstract. A new class of generalized uniformly Lipschitzian mappings is introduced, which includes the known class of uniformly Lipschitzian mappings in Banach spaces as special case. The convergence of the iterative approximation of fixed points for generalized uniformly Lipschitzian asymptotically pseudocontractive mappings by the modified Ishikawa iterative sequences with errors in real Banach sapces is proved. The results presented in this paper extend and improve some recent results by Goebel and Kirk, Schu, Liu, Zhang and others.

1. Introduction and Preliminaries

Let X be a real Banach space, X^*, the dual space of X, and J, the normalized duality mapping from X into 2^{X^*} defined by

$$J(x) = \{f \in X^* :< x, f >= \|f\|\|x\|, \|f\| = \|x\|\}, \forall x \in X,$$

where $< \cdot, \cdot >$ denotes the generalized duality pairing.

We know from Chang [1], X is uniformly smooth Banach spaces if and only if J is single-valued and J is uniformly continuous on bounded subsets of X. We denote the single-valued normalized duality mapping by j.

Definition 1.1. Let E be a nonempty subset of X, $T : E \to E$ is said to be *asymptotically pseudocontractive,* if there exists a real sequence $\{r_n\} \subset [1, \infty)$

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with $\lim_{n\to\infty} r_n = 1$, and for any $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$< T^n x - T^n y, j(x-y) > \leq r_n \|x-y\|^2, \forall n \geq 1.$$

Definition 1.2. Let E be a nonempty subset of X, $T : E \to E$ be a mapping, T is said to be *uniformly Lipschitzian,* if there exists some constant $c > 0$ such that

$$\|T^n x - T^n y\| \leq c\|x-y\|, \forall n \in N,$$

for all $x, y \in E$.

The conception of an asymptotically non-expansive mapping was introduced by Goebel and Kirk [3] in 1972, which was closely related to the theory of fixed points of mappings in Banach spaces. An early fundamental result, due to Goebel and Kirk [3], shows that if X is a uniformly convex Banach space, D is a nonempty bounded closed convex subset of X and $T : D \to D$ is an asymptotically non-expansive mapping, then T has a fixed point in D. On the other hand, the conception of asymptotically pseudocontractive mapping was introduced by Schu [5] in 1991.

The iterative approximation problems for asymptotically non-expansive and asymptotically pseudocontractive mappings were studied extensively by Chang [2], Goebel and Kirk [3], Liu [4] and Schu [5] in the setting of either Hilbert spaces or uniformly convex Banach spaces or Banach spaces.

Recently, Chang [2] proved the following theorem.

Theorem. *Let E be a nonempty convex subset of a Banach space X, $T : E \to E$ be a uniformly L-Lipschitz asymptotically pseudocontractive mapping with the sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$, $L \geq 1$. Let $\{\alpha_n\}, \{\beta_n\}$ be two real sequences in $[0,1]$ satisfying the following conditions:*

i) $\alpha_n \to 0, \beta_n \to 0 (n \to \infty)$;

ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For any $x_0 \in D$, let $\{x_n\}$ be the Ishikawa iterative sequence defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \forall n \geq 0. \end{cases}$$

If $F(T) \neq \phi$ and for any given $q \in F(T)$, there exists a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(0) = 0$ such that

$$< T^n x_{n+1} - q, j(x_{n+1} - q) > \leq k_n \|x_{n+1} - q\|^2 - \varphi(\|x_{n+1} - q\|), \forall n \geq 0,$$

where $j(x_{n+1} - q) \in J(x_{n+1} - q)$*, then the sequence* $\{x_n\}$ *converges strongly to the unique fixed point* q *of* T*.*

In this paper, a new class of generalized uniformly Lipschitzian mappings is introduced, which includes the known class of uniformly Lipschitzian mappings in Banach spaces as special case. The convergence of the iterative approximation of fixed points for generalized uniformly Lipschitzian asymptotically pseudocontractive mappings by the modified Ishikawa iterative sequences with errors in real Banach sapces is proved. The results presented in this paper extend and improve some recent results by Goebel and Kirk, Schu, Liu, Zhang and others.

For the purpose of this paper, we give some definitions and lemmas.

Definition 1.3. Let E be a nonempty subset of X, $T : E \to E$ be a mapping, T is said to be *generalized uniformly Lipschitzian,* if there exists some constant $c > 0$ such that

$$\|T^n x - T^n y\| \le c(1 + \|x - y\|),$$

for all $x, y \in E$ and $n = 1, 2, \cdots$.

Remark 1. If T is uniformly Lipschitzian, then T is generalized uniformly Lipschitzian; conversely if T is generalized uniformly Lipschitzian, then T may not uniformly Lipschitzian and even may not continuous.

Definition 1.4. Let E be a nonempty subset of X, $T : E \to E$ be a mapping and $x_0 \in E$ be a given point. The sequence $\{x_n\}$ defined by (ISE)

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^{k_n} y_n + u_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^{m_n} x_n + v_n, \forall n \ge 0, \end{cases}$$

is called the *modified Ishikawa iterative sequence with errors* of T, where $\{m_n\}$ and $\{k_n\}$ are two any positive integer sequences, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are two sequences in E and $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[0, 1]$ satisfying some additional conditions.

In particular, if $u_n = v_n = 0, \forall n \ge 0$, then the sequences $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^{k_n} y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^{m_n} x_n, \forall n \ge 0, \end{cases}$$

is called the *modified Ishikawa iterative sequence* of T.

If $\beta_n = 0, v_n = 0, \forall n \ge 0$, then the sequence $\{x_n\} \subset E$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^{k_n} x_n + u_n, \forall n \ge 0,$$

is called the *modified Mann iterative sequence with errors* of T.

Lemma 1.1. ([1]) *Let X be a real Banach space and $J : X \to 2^{X^*}$ be a normalized duality mapping, then for all $x, y \in X$*

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, j(x+y) >,$$

for all $j(x+y) \in J(x+y)$.

Lemma 1.2. ([6]) *Let $\{a_n\}, \{b_n\}, \{c_n\}, \{o(t_n)\}$ be nonnegative real sequences satisfying the condition*

$$a_{n+1} \leq (1-t_n)a_n + b_n a_n + c_n + o(t_n), \forall n \geq 0,$$

where $\{t_n\}$ is a sequence in $[0,1]$ with $\sum_{n=0}^{\infty} t_n = +\infty$, $\sum_{n=0}^{\infty} b_n < +\infty$ and $\sum_{n=0}^{\infty} c_n < +\infty$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main Results

Theorem 2.1. *Let X be a real uniformly smooth Banach spaces, E be a nonempty convex subset of X and $E + E \subset E$, $T : E \to E$ be a generalized uniformly Lipschitz asymptotically pseudocontractive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[0,1]$ satisfying $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = +\infty$; $\{m_n\}, \{k_n\}$ two any positive integer sequences satisfying $\lim_{n\to\infty} k_n = \infty$; $\{u_n\}, \{v_n\}$ two sequences in E satisfying the one of the following* (I) *or* (II)

(I) $\|u_n\| = o(\alpha_n), \|v_n\| \to 0 (n \to \infty)$;

(II) $\sum_{n=0}^{\infty} \|u_n\| < +\infty, \|v_n\| \to 0 (n \to \infty)$.

Let $\{x_n\} \subset E$ be the modified Ishikawa iterative sequence with errors defined by (ISE), *if $F(T) \neq \phi$ and for any given $q \in F(T)$, there exists a strictly increasing function $\varphi : [0,\infty) \to [0,\infty), \varphi(0) = 0$ such that $\forall x \in E$*

$$< T^n x - q, j(x-q) > \leq r_n \|x-q\|^2 - \varphi(\|x-q\|)\|x-q\|, \forall n \geq 0,$$

and if φ satisfies

$$\liminf_{x\to\infty} \frac{\varphi(x)}{x} = 2\sigma > 0,$$

then the sequence $\{x_n\}$ converges strongly to the unique fixed point q of T.

Proof. In the sequel, we denote the generalized uniformly Lipschitzian constant of T by c, without loss of generality, we assume that $c \geq 1$. By the condition of (I), let $\|u_n\| = d_n \alpha_n, \lim_{n\to\infty} d_n = 0$ (without loss of generality, let $d_n \in (0,1)$). Let $q_1, q \in F(T)$, then we have

$$< T^n q_1 - q, j(q_1 - q) > \leq r_n \|q_1 - q\|^2 - \varphi(\|q_1 - q\|)\|q_1 - q\|.$$

It follows from $\lim_{n\to\infty} r_n = 1$ that $q = q_1$, that is, q is the unique fixed point of T. By condition, let $\forall n \geq 0, \|u_n\| \leq 1, \|v_n\| \leq 1$.

Set $h_n = \|j(\frac{x_{n+1}-q}{1+\|x_n-q\|}) - j(\frac{y_n-q}{1+\|x_n-q\|})\|$.

By the definition of (ISE), we obtain

$$\begin{aligned}\|y_n - q\| &= \|(1-\beta_n)(x_n - q) + \beta_n(T^{m_n}x_n - q) + v_n\| \\ &\leq (1-\beta_n)\|x_n - q\| + \beta_n\|T^{m_n}x_n - q\| + \|v_n\| \\ &\leq (1+\beta_n c)\|x_n - q\| + \beta_n c + \|v_n\|,\end{aligned} \tag{1}$$

$$\begin{aligned}\|x_{n+1} - y_n\| &= \|(\beta_n - \alpha_n)x_n + \alpha_n T^{k_n}y_n - \beta_n T^{m_n}x_n + u_n - v_n\| \\ &= \|(\beta_n - \alpha_n)(x_n - q) + \alpha_n(T^{k_n}y_n - q) \\ &\quad - \beta_n(T^{m_n}x_n - q) + u_n - v_n\| \\ &\leq (\alpha_n + \beta_n)\|x_n - q\| + \alpha_n c(1 + \|y_n - q\|) + \beta_n c(1 + \|x_n - q\|) \\ &\quad + \|u_n\| + \|v_n\| \\ &\leq (3c^2\alpha_n + 2c\beta_n)\|x_n - q\| + 2c^2\alpha_n + \beta_n c \\ &\quad + (c+1)\|v_n\| + \|u_n\|,\end{aligned} \tag{2}$$

and

$$\begin{aligned}\frac{\|x_{n+1} - y_n\|}{1 + \|x_n - q\|} &\leq \frac{(3c^2\alpha_n + 2c\beta_n)(1 + \|x_n - q\|) + (c+1)\|v_n\| + \|u_n\|}{1 + \|x_n - q\|} \\ &\leq 3c^2\alpha_n + 2c\beta_n + (c+1)\|v_n\| + \|u_n\| \to 0 (n \to \infty).\end{aligned} \tag{3}$$

It follows from (ISE), (1) and inequality $2\|x_n - q\| \leq 1 + \|x_n - q\|^2$ that

$$\begin{aligned}\|x_{n+1} - q\| &\leq (1-\alpha_n)\|x_n - q\| + \alpha_n\|T^{k_n}y_n - T^{k_n}q\| + \|u_n\| \\ &\leq \|x_n - q\| + \alpha_n c[1 + (1+\beta_n c)\|x_n - q\| + \beta_n c + \|v_n\|] + \|u_n\| \\ &\leq 3c^2\|x_n - q\| + 4c^2 \\ &\leq \frac{3}{2}c^2\|x_n - q\|^2 + \frac{11}{2}c^2.\end{aligned} \tag{4}$$

Note that from (3)

$$\|\frac{(x_{n+1} - q) - (y_n - q)}{1 + \|x_n - q\|}\| \to 0 \ (n \to \infty).$$

It is easily seen that, in view of the uniform continuity of j on any bounded subset of X

$$\lim_{n\to\infty} h_n = 0.$$

So that

$$L_n = 8c^2\beta_n + 8c^2\|v_n\| + 6c^2 h_n \to 0 (n \to \infty).$$

By (1), we have

$$\frac{\|y_n - q\|}{1 + \|x_n - q\|} \le 1 + \beta_n c \le 2c, \tag{5}$$

$$\begin{aligned} \frac{\|T^{k_n} y_n - T^{k_n} q\|}{1 + \|x_n - q\|} &\le \frac{c(1 + \|y_n - q\|)}{1 + \|x_n - q\|} \\ &\le c(1 + 2c) \\ &\le 3c^2. \end{aligned} \tag{6}$$

Hence it follows from Lemma 1.1, (5) and (6) that

$$\begin{aligned} (\frac{\|y_n - q\|}{1 + \|x_n - q\|})^2 &= (\frac{\|(1-\beta_n)(x_n - q) + \beta_n(T^{m_n} x_n - q) + v_n\|}{1 + \|x_n - q\|})^2 \\ &\le (1-\beta_n)^2 \frac{\|x_n - q\|^2}{(1 + \|x_n - q\|)^2} \\ &\quad + 2\beta_n < \frac{T^{m_n} x_n - T^{m_n} q}{1 + \|x_n - q\|}, j(\frac{y_n - q}{1 + \|x_n - q\|}) > \\ &\quad + 2 < \frac{v_n}{1 + \|x_n - q\|}, j(\frac{y_n - q}{1 + \|x_n - q\|}) > \\ &\le \frac{\|x_n - q\|^2}{(1 + \|x_n - q\|)^2} + 2\beta_n c \|\frac{y_n - q}{1 + \|x_n - q\|}\| \\ &\quad + 2\|v_n\| \|\frac{y_n - q}{1 + \|x_n - q\|}\| \\ &\le \frac{\|x_n - q\|^2}{(1 + \|x_n - q\|)^2} + 2c(2\beta_n c + 2\|v_n\|), \end{aligned} \tag{7}$$

and

$$
\begin{aligned}
&\|\frac{x_{n+1}-q}{1+\|x_n-q\|}\|^2\\
&=\|(1-\alpha_n)\frac{x_n-q}{1+\|x_n-q\|}+\alpha_n\frac{T^{k_n}y_n-T^{k_n}q}{1+\|x_n-q\|}+\frac{u_n}{1+\|x_n-q\|}\|^2\\
&\le(1-\alpha_n)^2\frac{\|x_n-q\|^2}{(1+\|x_n-q\|)^2}\\
&\quad+2\alpha_n<\frac{T^{k_n}y_n-T^{k_n}q}{1+\|x_n-q\|},j(\frac{y_n-q}{1+\|x_n-q\|})>\\
&\quad+2\alpha_n<\frac{T^{k_n}y_n-T^{k_n}q}{1+\|x_n-q\|},j(\frac{x_{n+1}-q}{1+\|x_n-q\|})-j(\frac{y_n-q}{1+\|x_n-q\|})>\\
&\quad+2<\frac{u_n}{1+\|x_n-q\|},j(\frac{x_{n+1}-q}{1+\|x_n-q\|})>\\
&\le(1-\alpha_n)^2\frac{\|x_n-q\|^2}{(1+\|x_n-q\|)^2}+2\alpha_nr_{k_n}\frac{\|y_n-q\|^2}{(1+\|x_n-q\|)^2}\\
&\quad-2\alpha_n\frac{\varphi(\|y_n-q\|)\|y_n-q\|}{(1+\|x_n-q\|)^2}+2\alpha_nh_n\|\frac{T^{k_n}y_n-T^{k_n}q}{1+\|x_n-q\|}\|\\
&\quad+\frac{2\|u_n\|\|x_{n+1}-q\|}{(1+\|x_n-q\|)^2}\\
&\le(1-\alpha_n)^2\frac{\|x_n-q\|^2}{(1+\|x_n-q\|)^2}+2\alpha_nr_{k_n}\frac{\|x_n-q\|^2}{(1+\|x_n-q\|)^2}\\
&\quad+8\alpha_nr_{k_n}c(\beta_nc+\|v_n\|)-\frac{2\alpha_n\varphi(\|y_n-q\|)\|y_n-q\|}{(1+\|x_n-q\|)^2}\\
&\quad+2\alpha_nh_n(3c^2)+\frac{2\|u_n\|[\frac{3}{2}c^2\|x_n-q\|^2+\frac{11}{2}c^2]}{(1+\|x_n-q\|)^2}.
\end{aligned}
\tag{8}
$$

Which implies that

$$
\begin{aligned}
\|x_{n+1}-q\|^2&\le[\alpha_n^2+2\alpha_n(r_{k_n}-1)+1]\|x_n-q\|^2+\alpha_nr_{k_n}L_n(1+\|x_n-q\|)^2\\
&\quad-2\alpha_n\varphi(\|y_n-q\|)\|y_n-q\|+3c^2\|u_n\|\|x_n-q\|^2+11c^2\|u_n\|\\
&\le[\alpha_n^2+2\alpha_n(r_{k_n}-1)+1+2\alpha_nr_{k_n}L_n+3c^2\|u_n\|]\|x_n-q\|^2\\
&\quad-2\alpha_n\varphi(\|y_n-q\|)\|y_n-q\|+2\alpha_nr_{k_n}L_n+11c^2\|u_n\|.
\end{aligned}
\tag{9}
$$

Case 1 If $\{\|x_n-q\|\}$ is bounded, that is, there exists a constant $M>0, \forall n\ge 0$ such that $\|x_n-q\|\le M$. By the definition of (ISE) and the property of T , we know that $\{Tx_n\}, \{y_n\}, \{Ty_n\}$ are all bounded.

We now consider the following two cases.

Case 1.1 Assume that the condition (I) is satisfied. Then

$$\liminf_{n\to\infty}\{\|y_n - q\|\} = 2\delta = 0,.$$

In fact, if $\delta > 0$, without loss of generality, let $\forall n \geq 0, \|y_n - q\| \geq \delta > 0$, $\varphi(\|y_n - q\|) \geq \varphi(\delta) > 0$.

Observe that

$$\lim_{n\to\infty}[(\alpha_n + 2(r_{k_n} - 1) + 2r_{k_n}L_n + 3c^2d_n)M^2 + 2r_{k_n}L_n + 11c^2d_n] = 0,$$

then there exists $N_1 > 0$ such that $\forall n \geq N_1$, we have

$$(\alpha_n + 2(r_{k_n} - 1) + 2r_{k_n}L_n + 3c^2d_n)M^2 + 2r_{k_n}L_n + 11c^2d_n < \varphi(\delta)\delta.$$

So by (9), when $\forall n \geq N_1$, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 \leq{}& \|x_n - q\|^2 + [\alpha_n^2 + 2\alpha_n(r_{k_n} - 1) + 2\alpha_n r_{k_n}L_n + 3c^2\|u_n\|]M^2 \\
& - 2\alpha_n\varphi(\delta)\delta + 2\alpha_n r_{k_n}L_n + 11c^2\|u_n\| \\
\leq{}& \|x_n - q\|^2 + \alpha_n[(\alpha_n + 2(r_{k_n} - 1) + 2r_{k_n}L_n + 3c^2d_n)M^2 \\
& + 2r_{k_n}L_n + 11c^2d_n] - 2\alpha_n\varphi(\delta)\delta \\
\leq{}& \|x_n - q\|^2 - \alpha_n\varphi(\delta)\delta.
\end{aligned}$$

Hence

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n\varphi(\delta)\delta.$$

$$\begin{aligned}
\varphi(\delta)\delta(\sum_{n=N_1}^{\infty}\alpha_n) &\leq \sum_{n=N_1}^{\infty}[\|x_n - q\|^2 - \|x_{n+1} - q\|^2] \\
&\leq \|x_{N_1} - q\|^2 \leq M^2 < \infty,
\end{aligned}$$

which is contradict to the condition

$$\sum_{n=0}^{\infty}\alpha_n = +\infty.$$

Thus

$$\liminf_{n\to\infty}\{\|y_n - q\|\} = 0,$$

and hence there exists an infinite subsequence $\{y_{n_j}\}$ of the sequence $\{y_n\}$ such that

$$y_{n_j} \to q (j \to \infty).$$

From (ISE), we obtain

$$(1-\beta_{n_j})\|x_{n_j}-q\| \leq \|y_{n_j}-q\|+\beta_{n_j}\|T^{m_{n_j}}x_{n_j}-T^{m_{n_j}}q\|+\|v_{n_j}\| \to 0(j \to \infty),$$

thus there exists an infinite subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ such that

$$x_{n_j} \to q(j \to \infty).$$

So $\forall \epsilon \in (0,1)$, there exists a positive integer n_j such that $\|x_{n_j} - q\| < \epsilon$ and $\forall n \geq n_j$ we have

$$(\alpha_n + 2(r_{k_n} - 1) + 2r_{k_n}L_n + 3c^2 d_n)M^2 + 2r_{k_n}L_n + 11c^2 d_n < 2\varphi(\frac{\epsilon}{2})\frac{\epsilon}{2},$$

$$(\alpha_n + \beta_n)\|x_n\| + \alpha_n\|T^{k_n}y_n\| + \beta_n\|T^{m_n}x_n\| + \|u_n\| + \|v_n\| \leq \frac{\epsilon}{2}.$$

Next we prove that $\|x_{n_j+1} - q\| \leq \epsilon$, if it is not the case, we assume that

$$\|x_{n_j+1} - q\| > \epsilon.$$

By (ISE), we have

$$\begin{aligned}\epsilon < \|x_{n_j+1} - q\| &\leq \|x_{n_j+1} - y_{n_j}\| + \|y_{n_j} - q\| \\ &\leq (\alpha_{n_j} + \beta_{n_j})\|x_{n_j}\| + \alpha_{n_j}\|T^{k_{n_j}}y_{n_j}\| + \beta_{n_j}\|T^{m_{n_j}}x_{n_j}\| \\ &\quad + \|u_{n_j}\| + \|v_{n_j}\| + \|y_{n_j} - q\| \\ &\leq \frac{\epsilon}{2} + \|y_{n_j} - q\|,\end{aligned}$$

So that

$$\|y_{n_j} - q\| > \frac{\epsilon}{2},\ \varphi(\|y_{n_j} - q\|) > \varphi(\frac{\epsilon}{2}).$$

It follows from (9) that

$$\begin{aligned}\|x_{n_j+1} - q\|^2 &\leq \|x_{n_j} - q\|^2 + \alpha_{n_j}[(\alpha_{n_j} + 2(r_{k_{n_j}} - 1) + 2r_{k_{n_j}}L_{n_j} \\ &\quad + 3c^2 d_{n_j})M^2 + 2r_{k_{n_j}}L_{n_j} + 11c^2 d_{n_j}] - 2\alpha_{n_j}\varphi(\frac{\epsilon}{2})\frac{\epsilon}{2} \\ &\leq \|x_{n_j} - q\|^2 + \alpha_{n_j}[2\varphi(\frac{\epsilon}{2})\frac{\epsilon}{2}] - 2\alpha_{n_j}\varphi(\frac{\epsilon}{2})\frac{\epsilon}{2} \\ &= \|x_{n_j} - q\|^2,\end{aligned}$$

thus we have

$$\|x_{n_j+1} - q\| \leq \|x_{n_j} - q\| \leq \epsilon,$$

which is contradict to the assumption $\|x_{n_j+1} - q\| > \epsilon$. So

$$\|x_{n_j+1} - q\| \leq \epsilon.$$

By induction, we can prove that $\forall m \geq 1, \|x_{n_j+m} - q\| \leq \epsilon$. By the arbitrariness of $\epsilon \in (0,1)$, we know that $x_n \to q$ $(n \to \infty)$.

Case 1.2 Assume the condition (II) is satisfied. Similar to case 1.1, we can prove that $\liminf_{n\to\infty}\{\|y_n - q\|\} = 0$, thus there exists an infinite subsequences $\{y_{n_i}\}$ of the sequences $\{y_n\}$ such that

$$y_{n_i} \to q(i \to \infty).$$

By virtue of (ISE), there exists an infinite subsequences $\{x_{n_i}\}$ of the sequences $\{x_n\}$ such that

$$x_{n_i} \to q(i \to \infty).$$

Then $\forall \epsilon \in (0,1)$, there exists a positive integer n_i such that $\|x_{n_i} - q\| < \epsilon$ and $\forall n \geq n_i$, we have

$$\alpha_{n_i} + 2(r_{k_{n_i}} - 1) + 4r_{k_{n_i}} L_{n_i} < 2\varphi(\frac{\epsilon}{2})\frac{\epsilon}{2}, 14c^2 \sum_{n=n_i}^{\infty} \|u_n\| < \epsilon^2.$$

Similar to case 1.1, we can prove that

$$\|y_{n_i} - q\| > \frac{\epsilon}{2}, \varphi(\|y_{n_i} - q\|) > \varphi(\frac{\epsilon}{2}),$$

Combining this with (9), we have

$$\begin{aligned}
\|x_{n_i+1} - q\|^2 &\leq \|x_{n_i} - q\|^2 + \alpha_{n_i}(\alpha_{n_i} + 2(r_{k_{n_i}} - 1) + 2r_{k_{n_i}} L_{n_i})\epsilon^2 \\
&\quad + 3c^2\|u_{n_i}\|\epsilon^2 - 2\alpha_{n_i}\varphi(\frac{\epsilon}{2})\frac{\epsilon}{2} + 2\alpha_{n_i} r_{k_{n_i}} L_{n_i} + 11c^2\|u_{n_i}\| \\
&\leq \|x_{n_i} - q\|^2 + \alpha_{n_i}(\alpha_{n_i} + 2(r_{k_{n_i}} - 1) + 2r_{k_{n_i}} L_{n_i}) + 3c^2\|u_{n_i}\| \\
&\quad - 2\alpha_{n_i}\varphi(\frac{\epsilon}{2})\frac{\epsilon}{2} + 2\alpha_{n_i} r_{k_{n_i}} L_{n_i} + 11c^2\|u_{n_i}\| \\
&\leq \|x_{n_i} - q\|^2 + 14c^2\|u_{n_i}\| \\
&\leq \epsilon^2 + 14c^2\|u_{n_i}\|,
\end{aligned}$$

By induction, we can prove that $\forall m \geq 1$,

$$\begin{aligned}\|x_{n_i+m} - q\|^2 &\leq \epsilon^2 + 14c^2 \sum_{n=n_i}^{n_i+m-1} \|u_n\| \\ &\leq \epsilon^2 + \epsilon^2 \\ &= 2\epsilon^2.\end{aligned}$$

that is

$$x_n \to q (n \to \infty).$$

Case 2 If the sequence $\{\|x_n - q\|\}$ is not bounded, in this case, we assume that the conclusion is not hold, that is, $\lim_{n\to\infty} \|x_n - q\| \neq 0$.

Then from the case 1, we can obtain

$$\lim_{n\to\infty} \|x_n - q\| = +\infty. \tag{10}$$

and by (ISE), we get

$$(1 - \beta_n - \beta_n c)\|x_n - q\| \leq \|y_n - q\| + 1 + \beta_n c. \tag{11}$$

Thus

$$\lim_{n\to\infty} \|y_n - q\| = +\infty. \tag{12}$$

By (10)-(12), there exists a positive integer N_2, when $n \geq N_2$ we have

$$\frac{1}{2}\|x_n - q\|^2 \leq \|y_n - q\|^2. \tag{13}$$

Note that $\liminf_{x\to\infty} \frac{\varphi(x)}{x} = 2\sigma > 0$, without loss of generality, let $\forall n \geq 0$, $\frac{\varphi(\|y_n - q\|)}{\|y_n - q\|} \geq \sigma$ (let $\sigma \in (0, 1]$), thus

$$\varphi(\|y_n - q\|)\|y_n - q\| \geq \sigma\|y_n - q\|^2. \tag{14}$$

Substituting (13), (14) into (9) yields

$$\begin{aligned}\|x_{n+1} - q\|^2 \leq{}& [\alpha_n^2 + 2\alpha_n(r_{k_n} - 1) + 1 + 2\alpha_n r_{k_n} L_n + 3c^2\|u_n\|]\|x_n - q\|^2 \\ &- \alpha_n\sigma\|x_n - q\|^2 + 2\alpha_n r_{k_n} L_n + 11c^2\|u_n\|.\end{aligned} \tag{15}$$

We again consider the following two cases.

Case 2.1 If the condition (I) is satisfied, observe that

$$\lim_{n\to\infty}[\alpha_n + 2(r_{k_n} - 1) + 2r_{k_n}L_n + 3c^2 d_n] = \lim_{n\to\infty}(2r_{k_n}L_n + 11c^2 d_n) = 0,$$

then there exists a positive integer N_3 such that $\forall n \geq N_3$

$$\alpha_n + 2(r_{k_n} - 1) + 2r_{k_n}L_n + 3c^2 d_n < \frac{1}{2}\sigma.$$

Thus $\forall n \geq N_3$, by (15) we know

$$\|x_{n+1} - q\|^2 \leq (1 - \frac{1}{2}\sigma\alpha_n)\|x_n - q\|^2 + o(\alpha_n). \tag{16}$$

Case 2.2 If the condition (II) is satisfied, observe that

$$\lim_{n\to\infty}[\alpha_n + 2(r_{k_n} - 1) + 2r_{k_n}L_n] = 0,$$

then there exists a positive integer N_4 such that $\forall n \geq N_4$

$$\alpha_n + 2(r_{k_n} - 1) + 2r_{k_n}L_n < \frac{1}{2}\sigma.$$

Thus $\forall n \geq N_4$, by (15) we know

$$\|x_{n+1} - q\|^2 \leq (1 - \frac{1}{2}\sigma\alpha_n)\|x_n - q\|^2 + (3c^2\|u_n\|)\|x_n - q\|^2 + 11c^2\|u_n\| + o(\alpha_n). \tag{17}$$

From Lemma 1.2, (16) and (17), we obtain $x_n \to q(n \to \infty)$, which is a contradiction. Thus we have

$$\lim_{n\to\infty}\|x_n - q\| = 0.$$

This completes the proof. □

Remark 2. If the sequences $\{x_n\}$ is bounded in Theorem 2.1, then the assumption that $\liminf_{x\to\infty}\frac{\varphi(x)}{x} = 2\sigma > 0$ can be dropped.

Theorem 2.2. *Let X be a real uniformly smooth Banach spaces, E be a nonempty convex subset of X and $E+E \subset E$, $T : E \to E$ be a generalized uniformly Lipschitz asymptotically pseudocontractive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[0,1]$ satisfying $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = +\infty$; $\{m_n\}, \{k_n\}$ two any positive integer sequences satisfying $\lim_{n\to\infty} k_n = \infty$. Let $\{x_n\} \subset E$ be the modified Ishikawa iterative sequence defined by* (ISE), *if $F(T) \neq \phi$ and for any given $q \in F(T)$, there exists a strictly increasing function $\varphi : [0,\infty) \to [0,\infty), \varphi(0) = 0$ such that $\forall x \in E$*

$$< T^n x - q, j(x-q) >\leq r_n\|x-q\|^2 - \varphi(\|x-q\|)\|x-q\|, \forall n \geq 0,$$

and if φ satisfies

$$\liminf_{x\to\infty} \frac{\varphi(x)}{x} = 2\sigma > 0,$$

then the sequence $\{x_n\}$ converges strongly to the unique fixed point q of T.

Proof. Taking $u_n = 0$ and $v_n = 0$ for all $n \geq 0$ in Theorem 2.1, the conclusion can be obtained immediately. □

Theorem 2.3. *Let X be a real uniformly smooth Banach spaces, E be a nonempty convex subset of X and $E+E \subset E$, $T : E \to E$ be a generalized uniformly Lipschitz asymptotically pseudocontractive mapping. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = +\infty$; $\{m_n\}$ be any positive integer sequence ; $\{u_n\}$ be a sequence in E satisfying the one of the following* (I) *or* (II)

(I) $\|u_n\| = o(\alpha_n)$;

(II) $\sum_{n=0}^{\infty} \|u_n\| < +\infty$.

Let $\{x_n\} \subset E$ be the modified Mann iterative sequence with errors defined by (ISE), *if $F(T) \neq \phi$ and for any given $q \in F(T)$, there exists a strictly increasing function $\varphi : [0,\infty) \to [0,\infty), \varphi(0) = 0$ such that $\forall x \in E$*

$$< T^n x - q, j(x-q) >\leq r_n\|x-q\|^2 - \varphi(\|x-q\|)\|x-q\|, \forall n \geq 0,$$

and if φ satisfies

$$\liminf_{x\to\infty} \frac{\varphi(x)}{x} = 2\sigma > 0,$$

then the sequence $\{x_n\}$ converges strongly to the unique fixed point q of T.

Proof. Taking $\beta_n = 0$ and $v_n = 0$ for all $n \geq 0$ in Theorem 2.1, the conclusion can be obtained immediately. □

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