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NOTE ON THE EXPONENTIAL STABILITY OF C_0-SEMIGROUP

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Abstract. In this note, we give a necessary and sufficient chracterization of exponential stability of C_0 semigroups in Banach spaces, some known results can be derived from our theorem.

1. Introduction

Let X be a Banach space, a family of linear bonuded operators $\{T(t) : X \to X, t \in [0,\infty)\}$ is said to be a C_0 semigroup if it satisfies the following conditions

1. $T(0) = I$, $I-$ the identity operator on X,
2. $T(s+t) = T(s)T(t)$ for all $t, s \in [0,\infty)$,
3. $\lim_{t\to 0^+} \|T(t)x - x\| = 0$ for all $x \in X$.

If $\{T(t) : t \in [0,\infty)\}$ is a C_0 semigroup, we call the operator A, defined by

$$Ax = \lim_{h\to 0^+} \frac{T(h)x - x}{h}, \quad x \in D(A),$$

the infinitesimal generator of C_0 semigroup $\{T(t)\}$. See Pazy [2]. In the past decade, characterization of asympototical stablity of C_0 group has been well studied, see [1], [5] and the refences therein, but for exponential stablity of C_0 group, there is not much known work, see [3], [4], no well characterization of exponential stability of C_0 group has been given. In this note, we give a chracterization of exponential stability of C_0 semigroups in Banach spaces, and also we derive some known results from our theorem.

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2. Main Results

Theorem 1. *Let $A : D(A) \to E$ be the generator of a C_0 semigroup $T(t)$. Suppose $Re\lambda < 0$ for all $\lambda \in \sigma(A)$. Then $T(t)$ is exponential stable if and only if $T(t)$ does not have continuous spectrum λ with $|\lambda| \geq 1$.*

Proof. We only need to show that $r(T(1)) < 1$. Since $\sigma(T(1)) = \sigma_p(T(1)) \cup \sigma_r(T(1)) \cup \sigma_c(T(1))$ and $\sigma_c(T(1)) \cap \{\lambda : |\lambda| \geq 1\} = \emptyset$, we only need to show that $|\lambda| < 1$ for all $\lambda \in \sigma_p(T(1)) \cup \sigma_r(T(1))$.

For $\lambda \in \sigma_p(T(1)) \cup \sigma_r(T(1))$ with $\lambda \neq 0$, by the spectrum mapping theorem, we know that there exists $\lambda_0 \in \sigma_p(A) \cup \sigma_r(A)$ such that $\lambda = e^{\lambda_0}$. It follows that $|\lambda| = e^{Re\lambda_0}$. But $Re\lambda < 0$ for all $\lambda \in \sigma(A)$. So we have $|\lambda| < 1$. This completes the proof. □

From Theorem 1, one easily see the solution of the following differential system is exponential stable, which is also well known by Lyapunov Theorem

$$\begin{cases} x'(t) = -Ax(t), t \in R, \\ x(0) = x_0 \in R^N, \end{cases}$$

where $A : R^N \to R^N$ is a positvie matrix.

Corollary 1. *If $Re\lambda < 0$ for all $\lambda \in \sigma(A)$, and $T(t_0)$ is compact for some $T_0 > 0$, then $T(t)$ is exponential stable.*

Proof. By assumption, we know that $T(t)$ is compact for all $t \geq t_0$, so $T(1)$ can not have continuous spectrum λ with $|\lambda| = 1$. □

Corollary 2. *If $T(t)$ is asymptotically stable, then $T(t)$ is exponential stable if and only if $T(1)$ has no continuous spectrum λ with $|\lambda| = 1$.*

Proof. Since $T(t)$ is asymptotically stable, $Re\lambda \leq 0$ for all $\lambda \in \sigma(A)$. Because $e^{\sigma(A)} \subseteq T(1)$, so we must have $\lambda < 0$ for all $\lambda \in \sigma(A)$, in virtue of Theorem 1, we know that Corollary 2 is true. □

Definition 1. ([6]) Let $T(t)$ be a C_0 semigroup, and let Ω be a compact set, we say that Ω attracts unit sphere if $T(t)\partial B \subseteq \Omega$ for sufficiently large $t > 0$, where $\partial B = \{x : \|x\| = 1\}$.

The following result is known in [6], here we give a different proof.

Corollary 3. *If $T(t)$ is asymptotical stable and a compact subset Ω attracts unit sphere, then $T(t)$ is exponentially stable.*

Proof. Suppose there exists $\lambda \in \sigma(T(1))$ such that $|\lambda| = 1$. Then there exist $x_n \in E$ with $\|x_n\| = 1$ such that $T(1)^n x_n - \lambda^n x_n \to 0$ as $n \to \infty$.

Since the compact Ω attracs unit sphere, $T(1)^n x_n$ has a convergence subsequence, without loss of generality, we may assume that $T(1)^n x_n$ conveges to y_0, therefore x_n has a subsequence converges to y_0, we may still denote it by x_n.

Since $\|T(1)^n y_0 - \lambda^n y_0\| \leq \|T(1)^n\| \|y_0 - x_n\| + \|y_0 - x_n\|$, it follows that $T(1)^n y_0$ does not converge to 0, which is a contradiction. □

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