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COINCIDENCE AND COMMON FIXED POINTS OF NON-SELF HYBRID MAPPINGS IN METRICALLY CONVEX SPACES

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Abstract. Ahmed and Khan obtained some results on common fixed points for a pair of multi-valued and single valued mappings in metrically convex spaces which extends many known results. However the proofs of their results contain some errors. In this paper we rectify these results and prove some common fixed point theorems for a single valued and a pair of multi-valued non self mappings in metrically convex metric spaces. Our results generalize and extends the results of Ciric and Ume, Assad and Kirk, Assad, Itoh and Khan.

1. Introduction

The study of fixed point theorems for multi-valued mappings was initiated by Nadler [10] and Markin [9]. Subsequently a number of generalizations of Nadler's contraction principle were obtained by many authors. Recently non linear hybrid contractions, that is contraction types involving single-valued and multi-valued mappings have been studied by many authors. Sufficient conditions for the existence of fixed points of multi-valued mappings of a closed subset K of a complete metric space into a class of closed bounded subsets of X have been studied by many authors([3],[4],[7],[8],[11]). Ahmed and Khan [1] obtained some results on common fixed points for a pair of multi-valued and single valued mappings in metrically convex spaces which extends many known results. However the proofs of their results contain some errors. In this paper we have rectified and improved the results of Ahmed and

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Khan [1] in a more general setting. Our results also generalize, improves and extends the results of Ciric and Ume [6], Assad and Kirk [4], Assad [3], Itoh [7] and Khan [8].

2. Preliminaries.

Let (X, d) be a metric space and $CB(X)$ denote the family of nonempty closed and bounded subsets of X. For $A, B \in CB(X)$, let $H(A, B)$ denote the Hausdorff metric induced by d on $CB(X)$ i.e.

$$H(A,B) = \max\{(\sup D(a,B) : a \in A), (\sup D(A,b) : b \in B)\}$$

where $D(x, A) = \inf\{d(x, a) : a \in A\}$. $CB(X)$ is a metric space with the distance function H.
Mappings $f : X \to X$ and $S : X \to CB(X)$ are said to commute at a point $z \in X$ iff $fSz \subset Sfz$; f and S are said to commute on X iff f and S commute at every point of X. Weak commutativity, compatibility, weak compatibility and commutativity of two single valued self maps on a metric space are equivalent at their coincidence point, but commutativity of f and S at their coincidence point is more general than compatibility, weak compatibility and weak commutativity of f and S. (Refer [12])

3. Main Results

Ahmed and Khan [1] established the following result:

Theorem A. [1, Theorem 3.1]. *Let (X, d) be a metrically convex metric space and K a nonempty closed subset of X. Let T be a mapping of K into $CB(X)$ and f be a mapping of K into X such that*

(1) $H(Tx, Ty) \leq \alpha d(fx, fy) + \beta\{D(fx, Tx) + D(fy, Ty)\}$
$+\gamma\{D(fx, Ty) + D(fy, Tx)\}$

(2) *where* $\alpha, \beta, \gamma \geq 0$, $\dfrac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2} < 1$

(3) $\partial K \subseteq f(K)$, $T(K) \subseteq f(K)$, $f(x) \in K \Longrightarrow Tx \subseteq K$

(4) $\{T, f\}$ *is a weakly commuting pair and*

(5) f *is continuous on* K,

then there exists a point z in K such that $z = fz \in Tz$.

Now we show that [1 ,Theorem 3.1] admits a counter example.

Example 1. *Let $X = \mathbb{R}$ be endowed with usual metric. $K = [0, 1]$. $T : K \to CB(X)$ and $f : K \to X$ be such that $Tx = \{0, 1\}$ and $fx = 1 - x$ for all x in K. We see that f and T satisfies all conditions of* [1, *Theorem* 3.1]. *Although* 0 *and* 1 *are coincident points of f and T, there do not exist any z in K satisfying $z = fz \in Tz$.*

The main problem with [1, Theorem 3.1] is in their proof (page 282) where they have used the inequality $d(Tx_n, Tz) \leq H(Fx_n - 1, Fz)$ which is incorrect. Similar error is found in [2] also.

Throughout this paper, we assume X to be metrically convex metric space in the sense of Menger, that is, X has the property that for each x, y in X with $x \neq y$, there exists z in X, $x \neq z, y \neq z$ such that

$$d(x, z) + d(z, y) = d(x, y)$$

Further (Refer [4],[5]) if K is a closed subset of X and if $x \in K, y \notin K$, then there exist z in ∂K, such that

$$d(x, z) + d(z, y) = d(x, y)$$

In all that follows, $C(A, f)$ denotes the set of coincidence points of the mappings $A : X \to CL(X)$ and $f : X \to X$, i.e. $C(A, f) = \{u : fu \in Au\}$, $C(A, B, f)$ denotes the set of coincidence points of the mappings $A : X \to CL(X)$, $B : X \to CL(X)$ and $f : X \to X$, i.e. $C(A, B, f) = \{u : fu \in Au \bigcap Bu\}$, where $CL(X)$ denotes the family of closed subsets of X.

We now state the following lemma which will be used in our main theorem.

Lemma 1. ([6]) *If $A, B \in CB(X)$ and $a \in A$, then for any positive number $q < 1$, there exists b in B such that $qd(a, b) \leq H(A, B)$.*

Our main theorem is as follows.

Theorem 1. *Let (X, d) be a metrically convex metric space and K a nonempty closed subset of X. Let S, T be mappings of K into $CB(X)$ and f be a mapping of K into X such that*

(3.1) $H(Sx, Ty) \leq \alpha d(fx, fy)$
$$+\beta \max\{D(fx, Sx) + D(fy, Ty), D(fx, Ty) + D(fy, Sx)\}$$
$$+\gamma \max\{D(fx, Sx) + D(fy, Sx), D(fx, Ty) + D(fy, Ty)\}$$

(3.2) $\lambda = \alpha + 3(\beta + 2\gamma) + \alpha(\beta + 2\gamma) < 1, \quad \alpha, \beta, \gamma \geq 0,$

(3.3) $S(\partial K) \subseteq f(K)$, $T(\partial K) \subseteq f(K)$ *and* $\partial(f(K)) \subseteq f(\partial K)$, *and*

(3.4) $f(K)$ *is complete*

Then, (1) f, S *and* T *has a coincidence point.*
Further if $f\{C(S,T,f)\} \subset K$, *then*
(2) f, S *and* T *has a common fixed point* fu *provided* $f(fu) = fu$ *and* f *commutes with* S *and* T *at* $u \in C(S,T,f)$.

Proof. Construct the sequences $\{x_n\}$ in K and $\{y_n\}$, $\{z_n\}$ in X, as follows: For some arbitrary x_0 in ∂K choose x_1 in K such that $z_1 = y_1 = fx_1 \in Sx_0$. Let a be any fixed number such that $0 < a < 1/2$. Put

$$q = \lambda^a.$$

Then from (3.2), $q < 1$. By Lemma 1 we can choose $y_2 \in Tx_1$ such that

$$qd(y_1, y_2) \leq H(Sx_0, Tx_1)$$

If $y_2 \in f(K)$, put $z_2 = y_2 = fx_2$. If $y_2 \notin f(K)$ then choose $z_2 \in \partial(f(K))$ such that $d(z_1, z_2) + d(z_2, y_2) = d(z_1, y_2)$. By (3.3) there will exist x_2 in K such that $z_2 = f(x_2)$. Choose $y_3 \in Sx_2$ such that

$$qd(y_2, y_3) \leq H(Tx_1, Sx_2)$$

Continuing this process we can form sequences $\{z_n\}$ and $\{y_n\}$ in X and $\{x_n\}$ in K such that

(i) $y_n \in Sx_{n-1}$ if n is odd and $y_n \in Tx_{n-1}$ if n is even
(ii) $qd(y_n, y_{n+1}) \leq H(Sx_{n-1}, Tx_n)$ if n is odd and
$qd(y_n, y_{n+1}) \leq H(Tx_{n-1}, Sx_n)$ if n is even
(iii) $z_n = f(x_n)$ for all $n \in N$
(iv) $y_n = z_n$ if $y_n \in f(K)$ else $x_n \in K$ and
$d(z_{n-1}, z_n) + d(z_n, y_n) = d(z_{n-1}, y_n)$

Let $P = \{z_i \in \{z_n\} : z_i = y_i\}$, $Q = \{z_i \in \{z_n\} : z_i \neq y_i\}$.
Observe that if $z_n \in Q$ for some n, then z_{n-1} and z_{n+1} belong to P, as two consecutive terms of $\{z_n\}$ cannot be in Q.
We claim that sequence $\{z_n\}$ is a Cauchy sequence. We consider the following three cases.
Case 1. $z_n \in P$ and $z_{n+1} \in P$. If n is odd, then from (ii) and (2.3) we have

$$\begin{aligned} qd(z_n, z_{n+1}) = qd(y_n, y_{n+1}) &\leq H(Sx_{n-1}, Tx_n) \\ &\leq \alpha d(fx_{n-1}, fx_n) + \beta \max\{D(fx_{n-1}, Sx_{n-1}) + D(fx_n, Tx_n), \\ &\quad D(fx_{n-1}, Tx_n) + D(fx_n, Sx_{n-1})\} \\ &\quad + \gamma \max\{D(fx_{n-1}, Sx_{n-1}) + D(fx_n, Sx_{n-1}), \\ &\quad D(fx_{n-1}, Tx_n) + D(fx_n, Tx_n)\} \end{aligned}$$

$$\begin{aligned}
&\leq \alpha d(fx_{n-1}, fx_n) \\
&\quad + \beta \max\{d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_{n+1})\} \\
&\quad + \gamma \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_{n+1})\} \\
&\leq \alpha d(fx_{n-1}, fx_n) + \beta\{d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})\} \\
&\quad + \gamma\{d(fx_{n-1}, fx_n) + 2d(fx_n, fx_{n+1})\} \\
&\leq \alpha d(fx_{n-1}, fx_n) + \beta\{d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})\} \\
&\quad + \gamma\{2d(fx_{n-1}, fx_n) + 2d(fx_n, fx_{n+1})\}.
\end{aligned}$$

Hence using (iii) we get,

$$d(z_n, z_{n+1}) \leq [(\alpha + \beta + 2\gamma)/(q - \beta - 2\gamma)]d(z_{n-1}, z_n) \tag{1}$$

A similar inequality is obtained for an even n.

Case 2. $z_n \in P$ and $z_{n+1} \in Q$ then using (iv) we have

$$qd(z_n, z_{n+1}) \leq qd(z_n, y_{n+1}) = qd(y_n, y_{n+1}).$$

Proceeding as in Case 1, we have for odd and even n

$$d(z_n, z_{n+1}) \leq [(\alpha + \beta + 2\gamma)/(q - \beta - 2\gamma)]d(z_{n-1}, z_n) \tag{2}$$

Case 3. $z_n \in Q$ and $z_{n+1} \in P$, now we have

$$d(z_n, z_{n+1}) \leq d(z_n, y_n) + d(y_n, z_{n+1}) = d(z_n, y_n) + d(y_n, y_{n+1}).$$

Let n be odd. Then from (ii) and (3.1) we have

$$\begin{aligned}
&qd(z_n, z_{n+1}) \\
&\leq qd(z_n, y_n) + qd(y_n, y_{n+1}) \leq qd(z_n, y_n) + H(Sx_{n-1}, Tx_n) \\
&\leq qd(z_n, y_n) + \alpha d(fx_{n-1}, fx_n) \\
&\quad + \beta \max\{D(fx_{n-1}, Sx_{n-1}) + D(fx_n, Tx_n), D(fx_{n-1}, Tx_n) + D(fx_n, Sx_{n-1})\} \\
&\quad + \gamma \max(D(fx_{n-1}, Sx_{n-1}) + D(fx_n, Sx_{n-1}), D(fx_{n-1}, Tx_n) + D(fx_n, Tx_n)\} \\
&\leq qd(z_n, y_n) + \alpha d(fx_{n-1}, fx_n) \qquad (3) \\
&\quad + \beta \max\{d(fx_{n-1}, y_n) + d(fx_n, y_{n+1}), d(fx_{n-1}, fx_{n+1}) + d(fx_n, y_n)\} \\
&\quad + \gamma \max\{d(fx_{n-1}, y_n) + d(fx_n, y_n), d(fx_{n-1}, z_{n+1}) + d(fx_n, z_{n+1})\}
\end{aligned}$$

since $\alpha < \lambda^a = q$ and $d(z_n, y_n) + d(z_{n-1}, z_n) = d(z_{n-1}, y_n)$ we have

$$qd(z_n, y_n) + \alpha d(z_{n-1}, z_n) < qd(z_{n-1}, y_n). \tag{4}$$

Also by the triangle inequality,

$$\begin{aligned}d(fx_{n-1}, fx_{n+1}) + d(fx_n, y_n) &\leq d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}) + d(fx_n, y_n)\\ &= d(fx_{n-1}, y_n) + d(fx_n, fx_{n+1}) \qquad (5)\end{aligned}$$

and, using (iv) we have

$$\begin{aligned}d(z_{n-1}, z_{n+1}) + d(z_n, z_{n+1}) &\leq d(z_{n-1}, z_n) + 2d(z_n, z_{n+1})\\ &\leq d(z_{n-1}, y_n) + 2d(z_n, z_{n+1}) \qquad (6)\\ &\leq 2d(z_{n-1}, y_n) + 2d(z_n, z_{n+1})\end{aligned}$$

Thus we have, using (3), (4), (5) and (6)

$$d(z_n, z_{n+1}) \leq [(q + \beta + 2\gamma)/(q - \beta - 2\gamma)]d(z_{n-1}, y_n). \qquad (7)$$

A similar inequality is obtained for an even n.
Since $z_n \in Q$ we have $z_{n-1} \in P$, and hence proceeding as in Case 2 we get

$$d(z_{n-1}, y_n) \leq [(\alpha + \beta + 2\gamma)/(q - \beta - 2\gamma)]d(z_{n-2}, z_{n-1}). \qquad (8)$$

Hence, using (7) and (8)

$$d(z_n, z_{n+1}) \leq \frac{(\alpha + \beta + 2\gamma)(q + \beta + 2\gamma)}{(q - \beta - 2\gamma)(q - \beta - 2\gamma)} d(z_{n-2}, z_{n-1}).$$

Let

$$\begin{aligned}h &= \frac{(\alpha + \beta + 2\gamma)(q + \beta + 2\gamma)}{(q - \beta - 2\gamma)(q - \beta - 2\gamma)}\\ &= 1 + \frac{(\alpha + \beta + 2\gamma)(q + \beta + 2\gamma) + 2q\beta + 4q\gamma - 4\beta\gamma\beta^2 - q^2 - 4\gamma^2}{(q - \beta - 2\gamma)^2}\end{aligned}$$

Since $q = \lambda^a < 1$, we have

$$\begin{aligned}h &\leq 1 + \frac{(\alpha + \beta + 2\gamma)(q + \beta + 2\gamma) + 2q\beta + 4q\gamma - 4\beta\gamma\beta^2 - q^2 - 4\gamma^2}{(q - \beta - 2\gamma)^2}\\ &= 1 - \frac{q^2 - (\alpha + 3(\beta + 2\gamma) + \alpha(\beta + 2\gamma))}{(q - \beta - 2\gamma)^2}.\end{aligned}$$

Since $\lambda^a > \lambda^{2a} > l$, by (3.2) we conclude that $h < 1$

Thus we see that in all cases

$$d(z_n, z_{n+1}) \leq h \max\{(d(z_{n-2}, z_{n-1}), d(z_{n-1}, z_n)\}, \quad \text{for all } n \geq 2.$$

Hence we have

$$d(z_n, z_{n+1}) \leq h^{\frac{n-1}{2}} \max\{(d(z_0, x_1), d(z_1, z_2)\},$$

For $m > n > N$, we get

$$d(z_n, z_m) \leq \sum_{i=N}^{\infty} d(z_i, z_{i+1}) \leq \frac{h^{N/2}}{h^{1/2} - h} \max\{d(z_0, z_1), d(z_1, z_2)\},$$

Thus $\{z_n\}$ is a Cauchy sequence. Since $f(K)$ is complete, the sequence $\{z_n\}$ being contained in $f(K)$ has a limit say z in $f(K)$. Let $u \in f^{-1}z$. Thus there exist u in K such that $fu = z$. From the way in which the $\{z_n\}$ is constructed, there exists an infinite subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\{z_{n_k}\} \in P$. Then for an odd $n_k = m$, we have

$$\begin{aligned}
D(z_{n_k}, T_u) &\leq H(Sx_{m-1}, T_u) \\
&\leq \alpha d(fx_{m-1}, f_u) + \beta \max\{(D(fx_{m-1}, Sx_{m-1}) + D(f_u, T_u), \\
&\quad D(fx_{m-1}, T_u) + D(f_u, Sx_{m-1})\} + \gamma \max\{(D(fx_{m-1}, Sx_{m-1}) \\
&\quad + D(f_u, Sx_{m-1}), D(fx_{m-1}, T_u) + D(f_u, T_u)\} \\
&\leq \alpha d(fx_{m-1}, f_u) + \beta \max\{(D(fx_{m-1}, fx_m) + D(f_u, T_u), \\
&\quad D(fx_{m-1}, T_u) + D(f_u, fx_m)\} \\
&\quad + \gamma \max\{(D(fx_{m-1}, fx_m) + D(f_u, fx_m), D(fx_{m-1}, T_u) + D(f_u, T_u)\}.
\end{aligned}$$

Taking the limit as $k \to \infty$ yields

$$D(fu, Tu) \leq (\beta + 2\gamma).D(fu, Tu),$$

which implies, as $\beta + 2\gamma < 1$ that $D(fu, Tu) = 0$. Since Tu is closed, $fu \in Tu$. Similarly, we can show that $fu \in Su$.
From (3.1)

$$\begin{aligned}
H(Su, Tu) &\leq \alpha d(fu, fu) \\
&\quad + \beta \max\{D(fu, Su) + D(fu, Tu), D(fu, Tu) + D(fu, Su)\} \\
&\quad + \gamma \max\{D(fu, Su) + D(fu, Su), D(fu, Tu) + D(fu, Tu)\} = 0,
\end{aligned}$$

Which implies $Su = Tu$. Thus u is a coincidence point of f, S and T.
To prove (2), let $w = f(u)$. Then using commutativity of f and S at u we have $w = fu = f(fu) \in f(Su) \subseteq S(fu) = Sw$. Thus w is a common fixed point of f and S. Similar argument yields that w is a common fixed point of f and T. □

We now apply Theorem 1 to prove the following:

Theorem 2. *Let (X, d) be a metrically convex metric space and K a nonempty closed subset of X. Let S, T be mappings of K into $CB(X)$ and f be a mapping of K into X such that* (3.3), (3.4) *and the following hold*;

$$\text{(3.5)}\quad H(Sx, Ty) \leq \alpha d(fx, fy) + \beta(D(fx, Sx) + D(fy, Ty)) + \gamma(D(fx, Ty) + D(fy, Sx)) + \delta(D(fx, Sx) + D(fy, Sx)) + \eta(D(fx, Ty) + D(fy, Ty)),$$

$$\text{(3.6)}\quad \text{where } \alpha, \beta, \gamma, \delta, \eta \geq 0,\ \frac{(\alpha+\beta+\gamma + 2\delta + 2\eta)(1 + \beta + \gamma + 2\delta + 2\eta)}{(1 - \beta - \gamma - 2\delta - 2\eta)^2} < 1$$

Then, (1) *f, S and T has a coincidence point.*
Further if $f\{C(S, T, f)\} \subset K$, then
(2) *f, S and T has a common fixed point fu provided $f(fu) = fu$ and f commutes with S and T at $u \in C(S, T, f)$.*

Proof. It is clear that (3.5) implies

$$\begin{aligned} H(Sx, Ty) \leq{} & \alpha d(fx, fy) \\ & +(\beta+\gamma)\max\{D(fx, Sx)+D(fy, Ty)), (D(fx, Ty)+D(fy, Sx)) \\ & +(\delta+\eta)\max\{D(fx, Sx)+D(fy, Sx)), (D(fx, Ty)+D(fy, Ty)) \end{aligned}$$

Also, (3.6) implies

$$\lambda = \alpha + 3(\beta + \gamma + 2(\delta + \eta)) + \alpha(\beta + \gamma + 2(\delta + \eta)) < 1,$$

Thus we see that all assumptions of Theorem 1 (with $\beta + \gamma$ instead of $\beta$and $\delta + \eta$ instead of γ) are fulfilled. □

Theorem 3. *Let (X, d) be a metrically convex metric space, K a nonempty closed subset of X and let $F = \{T_j\}_j \in J$ be a family of multi-valued mapping of K into $CB(X)$ and f be a mapping of K into X. Suppose that there exists some $T_i \in F$ such that for each $T_j \in F$*

$$\begin{aligned} \text{(3.7)}\quad H(T_ix, T_jy) \leq{} & \alpha_j d(fx, fy) \\ & +\beta_j \max\{D(fx, T_ix)+D(fy, T_jy), D(fx, T_jy)+D(fy, T_ix)\} \\ & +\gamma_j \max\{D(fx, T_ix)+D(fy, T_ix), D(fx, T_jy)+D(fy, T_jy)\} \end{aligned}$$

Where $\alpha_j, \beta_j, \gamma_j \geq 0$ are such that

(3.8) $\lambda_j = \alpha_j + 3(\beta_j + 2\gamma_j) + \alpha_j(\beta_j + 2\gamma_j) < 1$

If in addition for each $T_j \in F$

(3.9) $T_j(\partial K) \subseteq f(K)$ *and* $\partial(f(K)) \subseteq f(\partial K)$
(3.10) $f(K)$ *is complete*
Then (1) *there exists some $u \in K$, such that $fu \in T_ju$ for all $T_j \in F$.*
Further if $f\{C(T_j, f)\} \in K$, then

(2) *f, and T_j has a common fixed point fu provided $f(fu) = fu$ and f commutes with T_j at $u \in C(T_j, f)$ for all T_j in F.*

Proof. Let T_{j0} be an arbitrary, fixed member of F. Then from Theorem 1 with $S = T_i$ and $T = T_{j0}$, there will exist u in K such that $fu \in T_{j0}u = T_i u$. Let $T_j \in F$, $Tj \neq T_{j0}$ be arbitrary. Then from (3.7) we have

$$\begin{aligned} D(fu, T_j u) \le H(T_i u, T_j u) &\le \alpha_j d(fu, fu) \\ &+ \beta_j \max\{D(fu, T_i u) + D(fu, T_j u), D(fu, T_j u) + D(fu, T_i u) \\ &+ \gamma_j \max\{D(fu, T_i u) + D(fu, T_i u), D(fu, T_j u) + D(fu, T_j u)\} \end{aligned}$$

And hence

$$(1 - \beta_j - 2\gamma_j)D(fu, T_j u) \le 0.$$

Since $1 - \beta_j - 2\gamma_j < 1$ we have $D(fu, T_j u) = 0$, hence $fu \in T_j u$, which completes the proof.
To prove (2), let $w = f(u)$. Then using commutativity of f and T_j at u we have $w = f(w) = f(fu) \in f(T_j u) \subseteq T_j(fu) = T_j w$. Thus w is a common fixed point of f and T_j. □

In view of Example 1, we see that [1, Theorem 3.3] is also incorrect. With a view of giving a corrected and improved versions of [1 Theorem 3.3], we now present the following results:

Theorem 4. *Let (X, d) be a metrically convex metric space and K a nonempty compact subset of X. Let S, T be continuous mappings of K into $CB(X)$ and f be a continuous mapping of K into X such that* (3.3), (3.4) *and the following hold*:

(3.11) $$\begin{aligned} H(Sx, Ty) < \alpha d(fx, fy) & \\ &+\beta \max\{D(fx, Sx) + D(fy, Ty), D(fx, Ty) + D(fy, Sx)\} \\ &+\gamma \max\{D(fx, Sx) + D(fy, Sx), D(fx, Ty) + D(fy, Ty)\} \end{aligned}$$

(3.12) *where* $\lambda = \alpha + 3(\beta + 2\gamma) + \alpha(\beta + 2\gamma) = 1$, $\alpha, \beta, \gamma \ge 0$

Then (1) *f, S and T has a coincidence point.*
Further if $f\{C(S, T, f)\} \in K$, then
(2) *f, S and T has a common fixed point fu provided $f(fu) = fu$ and f commutes with S and T at $u \in C(S, T, f)$.*

Proof. Define $g_1(x) = D(fx, Sx)$ and $g_2(x) = D(fx, Tx)$ for all $x \in K$. Since for all $x, y \in K$
$D(fx, Sx) = d(fx, fy) + D(fy, Sx)$ and $D(fy, Sx) = D(fy, Sy) + H(Sy, Sx)$

We have

$$\begin{aligned}|g_1(x) - g_1(y)| &= |D(fx, Sx) - D(fy, Sx)| + |D(fy, Sx) - D(fy, Sy)| \\ &= d(fx, fy) + H(Sx, Sy).\end{aligned}$$

Since f and S are continuous, $g_1(x)$ is continuous. Similarly it can be shown that $g_2(x)$ is continuous. Hence the function $h(x) = \min\{g_1(x), g_2(x)\}$ is continuous, and since K is compact, there exists $z \in K$ such that $h(z) = \min\{h(x) : x \in K\}$. Suppose $h(z) = D(fz, Sz)$. i.e.

$$D(fz, Sz) \leq \min\{D(fx, Sx), D(fx, Tx)\} \text{ for all } x \in K.$$

We claim that $D(fz, Sz) = 0$. Suppose not. Then $h(x) > 0$ for all $x \in K$. Consider the sequence $\{z_n\}$ in Sz, such that

$$Lim_{n\to\infty} d(fz, zn) = D(fz, Sz). \tag{9}$$

Suppose there exists an infinite subsequence $\{z_{n_k}\}$ of $\{z_n\}$ which is contained in $f(K)$. Then since $f(K)$ is compact $\{z_{n_k}\}$ will converge to some $z_0 = f(x_0)$. Since Sz is closed, $f(x_0) \in Sz$. Thus $d(fz, fx_0 = D(fz, Sz)$. From (3.11) we have,

$$\begin{aligned}D(fx_0, Tx_0) &= H(Sz, Tx_0) \\ &< \alpha d(fz, fx_0) + \beta \max\{D(fz, Sz) + D(fx_0, Tx_0), D(fz, Tx_0)\} \\ &\quad + \gamma \max\{D(fz, Sz), D(fz, Tx_0) + D(fx_0, Tx_0)\}\end{aligned}$$

Now since $D(fz, Tx_0) = d(fz, fx_0) + D(fx_0, Tx_0) = D(fz, Sz) + D(fx_0, Tx_0)$ we have

$$\begin{aligned}D(fx_0, Tx_0) < \alpha d(fz, Sz) &+ \beta\{D(fz, Sz) + D(fx_0, Tx_0) \\ &+ \gamma\{D(fz, Sz) + 2D(fx_0, Tx_0)\}\end{aligned}$$

Since $D(fz, Sz) = D(fx_0, Tx_0)$, we have

$$D(fx_0, Tx_0) = (\alpha + 2\beta + 3\gamma) D(fx_0, Tx_0) < D(fx_0, Tx_0),$$

a contradiction.
Now suppose $z_n \notin f(K)$ for all sufficiently large n. Then since X is convex and $fz \in f(K)$, for each z_n there exists $y_n \in \partial f(K)$ such that

$$d(fz, y_n) + d(y_n, z_n) = d(fz, z_n). \tag{10}$$

By (3.3) there will exist $w_n \in \partial K$, such that $y_n = f(w_n)$.

Since ∂K is compact, let $\{w_n\}$ converge to some $w_0 \in \partial K$. Since g_2 is continuous,

$$\lim_{n\to\infty} D(fw_n, Tw_n) = D(fw_0, Tw_0). \tag{11}$$

By (10), (3.11) and triangle inequality, we have

$$\begin{aligned}
D(fw_n, Tw_n) &\le d(fw_n, z_n) + D(z_n, Tw_n)\\
&= d(fz, z_n) - d(fz, fw_n) + H(Sz, Tw_n)\\
&< d(fz, z_n) - d(fz, fw_n) + \alpha d(fz, fw_n)\\
&\quad +\beta \max\{D(fz, Sz)+D(fw_n, Tw_n), D(fz, Tw_n)+D(fw_n, Sz)\}\\
&\quad +\gamma \max\{D(fz, Sz)+D(fw_n, Sz), D(fz, Tw_n)+D(fw_n, Tw_n)\}\\
&\le d(fz, z_n) + \beta \max\{D(fz, Sz) + D(fw_n, Tw_n),\\
&\quad D(fz, fw_n) + D(fw_n, Tw_n) + D(fw_n, z_n)\}\\
&\quad + \gamma \max\{D(fz, Sz) + d(fw_n, fz) + D(fz, Sz),\\
&\quad d(fz, fw_n) + D(fw_n, Tw_n) + D(fw_n, Tw_n)\}\\
&\le d(fz, z_n)\\
&\quad +\beta \max\{D(fz, Sz)+D(fw_n, Tw_n), D(fz, z_n)+D(fw_n, Tw_n)\}\\
&\quad + \gamma \max\{2D(fz, Sz) + D(fz, z_n) + 2D(fw_n, Tw_n)\}
\end{aligned}$$

Taking the limit as n tends to infinity and considering (9) and (11), we get

$$\begin{aligned}
D(fw_0, Tw_0) \le D(fz, Sz) &+ \beta\{D(fz, Sz) + D(fw_0, Tw_0)\}\\
&+ \gamma\{2D(fz, Sz) + 2D(fw_0, Tw_0)\}.
\end{aligned}$$

Hence

$$D(fw_0, Tw_0) \le \frac{(1 + \beta + 2\gamma)}{(1 - \beta - 2\gamma)} D(fz, Sz) \tag{12}$$

Since $w_0 \in \partial K$, $Tw_0 \subseteq f(K)$. Thus Tw_0 is compact and so there exists $u \in Tw_0$ such that $d(fw_0, u) = D(fw_0, Tw_0)$. Let $u = fv$, for some $v \in K$. From (3.11),

$$\begin{aligned}
D(fv, Sv) &\le H(Tw_0, Sv)\\
&< \alpha d(fv, fw_0) + \beta \max\{D(fv, Sv) + D(fw_0, Tw_0), D(fw_0, Sv)\}\\
&\quad + \gamma \max\{D(fv, Sv) + D(fw_0, Sv), D(fw_0, Tw_0)\}
\end{aligned}$$

Since $D(fw_0, Sv) \leq d(fw_0, fv) + D(fv, Sv) = D(fw_0, Tw_0) + D(fv, Sv)$, we have

$$\begin{aligned} D(fv, Sv) < \alpha D(fw_0, Tw_0) + \beta\{D(fv, Sv) + D(fw_0, Tw_0)\} \\ + \gamma\{2D(fv, Sv) + 2D(fw_0, Tw_0)\} \end{aligned}$$

Hence

$$D(fv, Sv) < \frac{\alpha + \beta + 2\gamma}{1 - \beta - 2\gamma} D(fw_0, Tw_0).$$

So by (12) we have

$$D(fv, Sv) < \frac{(\alpha + \beta + 2\gamma)(1 - \beta - 2\gamma)}{(1 - \beta - 2\gamma)(1 - \beta - 2\gamma)} D(fz, Sz). \tag{13}$$

Since,

$$\frac{(\alpha + \beta + 2\gamma)}{(1 - \beta - 2\gamma)} \frac{(1 - \beta - 2\gamma)}{(1 - \beta - 2\gamma)} = \frac{\alpha + 3(\beta + 2\gamma) + \alpha(\beta + 2\gamma) - 2(\beta - 2\gamma) + (\beta - 2\gamma)^2}{1 - 2(\beta - 2\gamma) + (\beta - 2\gamma)^2}$$

by (3.12) we get

$$\frac{(\alpha + \beta + 2\gamma)}{(1 - \beta - 2\gamma)} \frac{(1 - \beta - 2\gamma)}{(1 - \beta - 2\gamma)} \leq 1.$$

Hence by (13), we have

$$D(fz, Sz) < D(fz, Sz),$$

a contradiction.
Hence $D(fz, Sz) = 0$. Since Sz is closed, we get $fz \in Sz$.
If $fz \notin Tz$, then by (3.11) we have

$$D(fz, Tz) \leq H(Sz, Tz) < (\beta + 2\gamma) D(fz, Tz),$$

a contradiction.
Thus z is a coincidence point of f, S and T. The remaining proof goes on the same lines as that of Theorem 1. $\square$

Theorem 5. *Let (X, d) be a metrically convex metric space and K a nonempty compact subset of X. Let S, T be continuous mappings of K into $CB(X)$ and f be a continuous mapping of K into X such that* (3.3), (3.4) *and the following hold*:

(3.13) $H(Sx,Ty) < \alpha d(fx,fy) + \beta\{D(fx,Sx) + D(fy,Ty)\}$
$+\gamma\{D(fx,Ty)+D(fy,Sx)\}+\delta\{D(fx,Sx)+D(fy,Sx)\}$
$+\eta\{D(fx,Ty) + D(fy,Ty)\}$

(3.14) *where* $\alpha,\beta,\gamma \geq 0$, $\dfrac{(\alpha+\beta+\gamma+2\delta+2\eta)(1+\beta+\gamma+2\delta+2\eta)}{(1-\beta-\gamma-\delta-2\eta)^2} \leq 1$

Then (1) f, S *and* T *has a coincidence point.*
Further if $f\{C(S,T,f)\} \in K$, *then*
(2) f, S *and* T *has a common fixed point* fu *provided* $f(fu) = fu$ *and* f *commutes with* S *and* T *at* $u \in C(S,T,f)$.

Proof. It is clear that (3.13) implies

$$
\begin{aligned}
H(Sx,Ty) < {} & \alpha d(fx,fy) \\
& +(\beta+\gamma)\max\{(D(fx,Sx)+D(fy,Ty)),D(fx,Ty)+D(fy,Sx)\} \\
& +(\delta+\eta)\max\{(D(fx,Sx)+D(fy,Sx)),D(fx,Ty)+D(fy,Ty)\}
\end{aligned}
$$

Since (3.14) implies

$$\lambda = \alpha + 3(\beta+\gamma+2(\delta+\eta)) + \alpha(\beta+\gamma+2(\delta+\eta)) \leq 1,$$

we see that all assumptions of Theorem 4 (with $\beta+\gamma$ instead of β and $\delta+\eta$ instead of γ) are fulfilled. Hence the proof follows from Theorem 4. □

Now we derive certain corollaries from our Theorems, which contain and improve many coincidence and fixed point theorems in convex metric spaces. Taking $\gamma = 0$, in Theorem 1 we have the following.

Corollary 1. *Let (X, d) be a metrically convex metric space and K a nonempty closed subset of X. Let S, T be mappings of K into $CB(X)$ and f be a mapping of K into X such that* (3.3), (3.4) *and the following holds*

(3.15) $H(Sx,Ty) \leq \alpha d(fx,fy)$
$+\beta\max\{D(fx,Sx) + D(fy,Ty), D(fx,Ty) + D(fy,Sx)\}$

(3.16) *where* $\alpha,\beta \geq 0$, $\lambda = \alpha + 3\beta + \alpha\beta < 1$

Then all conclusions of Theorem 1 *are true.*

Taking $\delta, \eta = 0$ in Theorem 2 we have the following:

Corollary 2. *Let (X,d) be a metrically convex metric space and K a non empty closed subset of X. Let S,T be mappings of K into $CB(X)$ and f be a mapping of K into X such that* (3.3)*,* (3.4) *and the following holds*

(3.17) $H(Sx,Ty) \leq \alpha d(fx,fy) + \beta(D(fx,Sx) + D(fy,Ty))$
$+\gamma(D(fx,Ty) + D(fy,Sx))$

(3.18) *where* $\alpha,\beta,\gamma \geq 0,\ \dfrac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2} < 1$

Then all conclusions of Theorem 1 *are true.*

Corollary 3. *Let (X,d) be a metrically convex metric space and K a non empty closed subset of X. Let T be a mapping of K into $CB(X)$ and f be a mapping of K into X such that* (3.3)*,* (3.4) *and the following holds*

(3.19) $H(Tx,Ty) \leq \alpha d(fx,fy) + \beta(D(fx,Tx) + D(fy,Ty))$
$+\gamma(D(fx,Ty) + D(fy,Tx))$

(3.20) *where* $\alpha,\beta,\gamma \geq 0,\ \dfrac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2} < 1$

Then f and T has a coincidence point.
Further if $fC(T,f) \subseteq K$, then
(2) *f, and T has a common fixed point fu provided $f(fu) = fu$ and f commutes with T at $u \in C(T,f)$.*

Proof. The proof follows by taking $S = T$ in Corollary 2. □

Taking $\gamma = 0$, in Theorem 3 we have the following:

Corollary 4. *Let (X,d) be a metrically convex metric space, K a nonempty closed subset of X and let $F = \{Tj\}_{jinj}$ be a family of multi-valued mapping of K into $CB(X)$ and f be a mapping of K into X. Suppose that there exists some $Ti \in F$ such that for each $Tj \in F$,* (3.10), (3.11), (3.12) *and the following holds*

(3.21) $H(Tix,Tjy) \leq \alpha_j d(fx,fy) + \beta_j \max\{(D(fx,T_ix) + D(fy,T_jy)),$
$(D(fx,T_jy) + D(fy,T_jx))\}$

Where $\alpha_j, \beta_j \geq 0$ are such that
(3.22) $\lambda_j = \alpha_j + 3\beta_j + \alpha_j\beta_j < 1$
Then all conclusions of Theorem 3 *are true.*

Taking $\delta,\eta = 0$ in Theorem 5 we have the following:

Corollary 5. *Let (X,d) be a metrically convex metric space and K a non empty compact subset of X. Let S,T be continuous mappings of K into $CB(X)$ and f a continuous mapping of K into X such that* (3.3)*,* (3.4) *and the following holds*

(3.23) $H(Sx,Ty) < \alpha d(fx,fy) + \beta(D(fx,Sx) + D(fy,Ty))$
$\quad + \gamma(D(fx,Ty) + D(fy,Sx))$

(3.24) *where* $\alpha,\beta,\gamma \geq 0,\ \dfrac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2} \leq 1$

Then all conclusions of Theorem 1 *are true.*

Corollary 6. *Let* (X,d) *be a metrically convex metric space and* K *a non empty compact subset of* X*. Let* T *be a continuous mapping of* K *into* $CB(X)$ *and* f *a continuous mapping of* K *into* X *such that* (3.3)*,* (3.4) *and the following holds*

(3.25) $H(Tx,Ty) < \alpha d(fx,fy) + \beta b(D(fx,Tx) + D(fy,Ty))$
$\quad + \gamma g(D(fx,Ty) + D(fy,Tx))$

(3.26) *where* $\alpha,\beta,\gamma \geq 0,\ \dfrac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2} \leq 1$

Then, (1) f *and* T *has a coincidence point.*
Further if $fC(T,f) \subseteq K$*, then*
(2) f*, and* T *has a common fixed point* fu *provided* $f(fu) = fu$ *and* f *commutes with* T *at* $u \in C(T,f)$*.*

Proof. The proof follows by taking $S = T$ in Corollary 5. □

Remark 1. *Corollary* 3 *and Corollary* 6 *are the corrected and improved versions of* [1, *Theorem* 3.1] *and* [1, *Theorem* 3.3] *respectively.*

Remark 2. *Corollary* 2 *and Corollary* 5 *are the corrected and improved versions of* [2, *Theorem* 3.1] *and* [2, *Theorem* 3.4] *respectively.*

Remark 3. *If* $f = idx$ (*the identity map of* X)*, then Corollary* 1, 2, 4 *and* 5 *reduces to Theorem* 2.1*, Corollary* 2.1 *Theorem* 2.2 *and Theorem* 2.3 *respectively of* [6]*. Moreover we require the completeness of* $f(K)$ *only instead of the completeness of the entire space* X *as used in* [6]*. Hence our result is a substantial generalization, extension and improvement of the results of Ciric and Ume* [6]*, Khan* [8]*, Assad* [3] *and Itoh* [7]*.*

Remark 4. *There exists many coincident point theorems for non self mappings. It seems that there do not exist fixed point results for hybrid non self mappings. Our results shows the existence of fixed points for hybrid mappings even if the mappings under consideration are not self maps.*

The following example supports our claims.

Example 2. *Let* $X = (0,\infty)$, $K = [1,2]$, $S,T : K \to CB(X)$ *and* $f : K \to X$ *be given by* $Sx = Tx = \{1, 3/2, 2\}$ *and* $fx = 3 - x$*. We see that all conditions* *of Theorem* 1 *are satisfied and* 3/2 *is a common fixed point of* f, S *and* T, *where as Theorem* 2.1 *of Ciric and Ume* [6] *is not applicable as* X *is not complete. Moreover, we see that the mappings* f *and* S *are not self mappings.*

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