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GENERALIZED DISCREPANCY PRINCIPLE AND ILL-POSED EQUATIONS INVOLVING ACCRETIVE OPERATORS

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Abstract. In this note, without the weak continuity for the dual mapping of the Banach spaces, the convergence rates of the regularized solution for nonlinear ill-posed equations involving accretive operators are established on the base of the general variant of discrepancy principle for the choice of regularization parameter.

1. Introduction

Let X be a real reflexive Banach space having the property of approximations (see [6]) and X^* be its dual space that is strictly convex. For the sake of simplicity, the norms of X and X^* will be denoted by the symbol $\|.\|$. We write $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let A be a m-accretive operator in X, i.e. (see [6])

i) $\langle A(x+h) - A(x), J(h)\rangle \geq 0, \quad \forall x, h \in X,$

where J is the dual mapping of X, the mapping from X onto X^* satisfies the condition

$$\langle J(x), x\rangle = \|J(x)\|\|x\|, \quad \|J(x)\| = \|x\|, \quad \forall x \in X,$$

and

ii) $R(A + \lambda I) = X$ for each $\lambda > 0$, where $R(A)$ denotes the range of A and I is the identity operator in X.

We are interested in solving the operator equation

$$A(x) = f, \quad f \in X, \tag{1.1}$$

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where A is an weakly continuous and m-accretive operator in X. Suppose that (1.1) has a unique solution denoted by x_0.

Without additional conditions on the structure of A, as strongly or uniformly accretive property, equation (1.1) is, in general, one of ill-posed problems. To solve (1.1), we have to use stable methods. A well known one is the Tikhonov regularization. Its operator version for ill-posed equations involving accretive operator has the form (see [1])

$$A(x) + \alpha(x - x_*) = f_\delta, \quad \|f_\delta - f\| \leq \delta \to 0, \tag{1.2}$$

where $\alpha > 0$ is the parameter of regularization, and x_* is some element in X ($x_* \neq x_0$). Since A is m-accretive, equation (1.2) has a unique solution x_α^δ for each fixed $\alpha > 0$ and $\delta > 0$. Moreover, from (1.1), (1.2) and the accretive property of A it is easy to obtain the estimate

$$\|x_\alpha^\delta - x_0\| \leq \|x_0 - x_*\| + \delta/\alpha. \tag{1.3}$$

In [1] it was also shown that the function $\rho(\alpha) = \alpha\|x_\alpha^\delta - x_*\|$ is continuous, monotone non-decreasing, and if A is continuous at x_* then

$$\lim_{\alpha\to 0}\rho(\alpha) = 0, \quad \lim_{\alpha\to+\infty}\rho(\alpha) = \|A(x_*) - f_\delta\|.$$

Further, on the base of the results in [2] and [4] it is not difficult to verify that if $\|A(x_*) - f_\delta\| > K\delta^p$, $K > 2$, $0 < p \leq 1$, then there exists at least a value $\overline{\alpha} = \alpha(\delta)$ such that $\|A(x_{\alpha(\delta)}^\delta) - f_\delta\| = K\delta^p$, and $(K-1)\delta^p/\alpha(\delta) \leq 2\|x_0 - x_*\|$. Consequently, for the case $0 < p < 1$ we have $\delta/\alpha(\delta) \leq 2\|x_0 - x_*\|\delta^{1-p}/(K - 1) \to 0$, as $\delta \to 0$. Hence, if J is continuous and sequential weak continuous, then $x_{\alpha(\delta)}^\delta \to x_0$ (see [1], [4]). Unfortunately, the class of infinite-dimensional Banach spaces having J with these properties is very small (only l_p). It is natural to ask when the algorithm (1.2) can be applied for other Banach spaces.

In this paper, we show that the sequential weak continuous property of J will be redundant, when A is weakly continuous and

$$\|A(y) - A(x_0) - QA'(x_0)^*J(y - x_0)\| \leq \tau\|A(y) - A(x_0)\|, \quad \forall y \in X, \tag{1.4}$$

where τ is some positive constant, and Q is the dual mapping of X^*. Condition (1.4) for compact operator A in Hilbert space X was proposed in [5] for studying the similar issues. Moreover, in this paper instead of the discrepancy principle, we show that $\alpha = \alpha(\delta)$ can be chose by the more general form of the principle that is

$$\rho(\alpha) = K\delta^p\alpha^{-q}, \quad 0 < p \leq q. \tag{1.5}$$

The generalized discrepancy principle was first proposed in [3] for investigating convergence rates for linear ill-posed problems.

2. Main results

To obtain the result on the convergence rate for $\{x^\delta_{\alpha(\delta)}\}$ we need the following lemmas.

Lemma 1. *For each $p, q, \delta > 0$, there exists at least a value $\alpha > 0$ such that*

$$\rho(\alpha) = K\delta^p\alpha^{-q}.$$

Proof. It follows from [1] that $\alpha \to \rho(\alpha) = \alpha\|x^\delta_\alpha - x_*\|$ is continuous in $[\alpha_0, +\infty)$, $\alpha_0 > 0$, increasing, $\rho(\alpha) \to 0$ as $\alpha \to 0$, and goes to $\|A(x_*) - f_\delta\|$ as $\alpha \to +\infty$. Therefore, $\alpha \to \alpha^{1+q}\|x^\delta_\alpha - x_*\| = \alpha^q\rho(\alpha)$ possesses the same properties, excepting one, i.e., it goes to $+\infty$ as $\alpha \to +\infty$. Hence, the result follows from the intermediate value theorem. □

Lemma 2. $\lim_{\delta\to 0}\alpha(\delta) = 0$.

Proof. Let $\delta_n \to 0$, and $\alpha_n = \alpha(\delta_n) \to +\infty$ as $n \to \infty$. Since

$$A(x^{\delta_n}_{\alpha_n}) + \alpha_n(x^{\delta_n}_{\alpha_n} - x_*) = f_{\delta_n}$$

we have

$$\|x^{\delta_n}_{\alpha_n} - x_*\| \le \|A(x_*) - f_{\delta_n}\|/\alpha_n \to 0,$$

as $n \to \infty$. Therefore, $x^{\delta_n}_{\alpha_n} \to x_*$ as $n \to \infty$. On the other hand, $\|A(x^{\delta_n}_{\alpha_n}) - f_{\delta_n}\| = \rho(\alpha_n) = K\delta_n^p\alpha_n^{-q} \to 0$, as $n \to \infty$. It means that $A(x_*) = f$, i.e. x_* is a solution of (1.1). It contradicts that x_* is not a solution of (1.1).

Thus, $\alpha(\delta)$ remains bounded as $\delta \to 0$. Let $\delta_n \to 0$ as $n \to \infty$, and meantime $\alpha_n \to c > 0$. Then,

$$K\delta_n^p\alpha_n^{-q} = \alpha_n\|x^{\delta_n}_{\alpha_n} - x_*\| \quad \text{or} \quad \alpha_n^{1+q}\|x^{\delta_n}_{\alpha_n} - x_*\| = K\delta_n^p.$$

Consequently, $x^{\delta_n}_{\alpha_n} \to x_*$ as $n \to \infty$. Again, since

$$\alpha_n\|x^{\delta_n}_{\alpha_n} - x_*\| = \|f_{\delta_n} - A(x^{\delta_n}_{\alpha_n})\|$$

we have $A(x_*) = f$. Hence, $\lim_{\delta\to 0}\alpha(\delta) = 0$. □

Lemma 3. *If $q \ge p$, then* $\lim_{\delta\to 0}\delta/\alpha(\delta) = 0$.

Proof. It is easy to see that

$$K\left[\frac{\delta}{\alpha(\delta)}\right]^p = K[\delta^p\alpha(\delta)^{-q}]\alpha(\delta)^{q-p} = \rho(\alpha(\delta))\alpha(\delta)^{q-p}.$$

On the other hand, from (1.2) and (1.3) it follows

$$\begin{aligned}\|A(x^\delta_{\alpha(\delta)}) - f_\delta\| &= \alpha(\delta)\|x^\delta_{\alpha(\delta)} - x_*\| \\ &\le 2\alpha(\delta)\|x_0 - x_*\| + \delta \to 0\end{aligned}$$

as $\delta \to 0$. Therefore,

$$\lim_{\delta\to 0}\left[\frac{\delta}{\alpha(\delta)}\right]^p = 0.$$

The lemma is proved. $\square$

Lemma 4. *Let $0 < p \le q$. Then, there exist constants $C_1, C_2 > 0$ such that, for sufficiently small $\delta > 0$, the relation*

$$C_1 \le \delta^p \alpha(\delta)^{-1-q} \le C_2$$

holds.

Proof. Since

$$\begin{aligned}K\delta^p\alpha(\delta)^{-q-1} &= \alpha(\delta)^{-1}\rho(\alpha(\delta)) \\ &= \|x^\delta_{\alpha(\delta)} - x_*\| \le 2\|x_0 - x_*\| + \delta/\alpha(\delta)\end{aligned}$$

there exists a positive constant C_2 such that $\delta^p\alpha(\delta)^{-1-q} \le C_2$.

As X is reflexive and $\{x^\delta_{\alpha(\delta)}\}$ is bounded, there exists a subsequence of $\{x^\delta_{\alpha(\delta)}\}$ converging weakly to some element $\tilde{x}_* \in X$ such that

$$\|\tilde{x}_* - x_*\| \le \varliminf_{\delta\to 0}\|x^\delta_{\alpha(\delta)} - x_*\|.$$

Obviously, that $\tilde{x}_* \ne x_*$. Indeed, if $\tilde{x}_* = x_*$, then from (1.2) and the weak continuous property of A it follows $A(x_*) = f$. This fact contradicts $x_* \ne x_0$, again. Hence, there exists a positive constant C_1 such that $C_1 \le \delta^p\alpha(\delta)^{-1-q}$. $\square$

Theorem 2.1. *Assume that the following conditions hold:*

(i) *A is Fréchet differentiable with* (1.4),

(ii) *there exists an element $z \in X$ such that*

$$A'(x_0)z = x_* - x_0, \text{ and}$$

(iii) *the parameter $\alpha = \alpha(\delta)$ is chosen by* (1.5).

Then, we have

$$\|x^\delta_{\alpha(\delta)} - x_0\| = O(\delta^\theta),$$
$$\theta = \min\Big\{1 - \eta_1, \eta_2/2\Big\},$$
$$\eta_1 = \frac{p}{1+q},$$
$$\eta_2 = \begin{cases} \eta_1, & p \le 1 \\ 1 - pq/(1+q) & 1 < p < 1 + 1/q. \end{cases}$$

Proof. From (1.1)-(1.2) and condition (ii) of the theorem it follows

$$\begin{aligned} \|x^\delta_\alpha - x_0\|^2 &\le \frac{\delta}{\alpha}\|x^\delta_\alpha - x_0\| + \langle x_* - x_0, J(x^\delta_\alpha - x_0)\rangle \\ &\le \frac{\delta}{\alpha}\|x^\delta_\alpha - x_0\| + \langle z, A'(x_0)^* J(x^\delta_\alpha - x_0)\rangle. \end{aligned} \tag{2.1}$$

Further,

$$\begin{aligned} \|A'(x_0)^* J(x^\delta_\alpha - x_0)\| &= \|QA'(x_0)^* J(x^\delta_\alpha - x_0)\| \\ &\le \|A(x^\delta_\alpha) - A(x_0) - QA'(x_0)^* J(x^\delta_\alpha - x_0)\| + \|A(x^\delta_\alpha) - f\| \\ &\le (\tau + 1)\|A(x^\delta_\alpha) - f\| \\ &\le (\tau + 1)(\|A(x^\delta_\alpha) - f_\delta\| + \delta). \end{aligned} \tag{2.2}$$

If α is chosen by the general variant of the discrepancy principle (1.5), then from (2.1), (2.2) it implies that

$$\|x^\delta_{\alpha(\delta)} - x_0\|^2 \le \frac{\delta}{\alpha(\delta)}\|x^\delta_{\alpha(\delta)} - x_0\| + (\tau + 1)\|z\|(K\delta^p\alpha^{-q}(\delta) + \delta).$$

Consequently, from Lemma 4 we obtain

$$\begin{aligned} \|x^\delta_{\alpha(\delta)} - x_0\|^2 &\le C_2\delta^{1-p}\alpha(\delta)^q\|x^\delta_{\alpha(\delta)} - x_0\| + (\tau + 1)(K + 1)\|z\|\delta^p\alpha(\delta)^{-q} \\ &\le C_2C_1^{-q/(1+q)}\delta^{1-p/(1+q)}\|x^\delta_{\alpha(\delta)} - x_0\| \\ &\quad + C_2C_1^{-1/(1+q)}(\tau + 1)(K + 1)\|z\|\delta^{p/(1+q)}, \end{aligned}$$

for the case $p \le 1$. In the case $p > 1$, the second term of the right-hand side in the last inequality is replaced by

$$C_2C_1^{-1/(1+q)}(\tau + 1)(K + 1)\|z\|\delta^{1-pq/(1+q)}.$$

Using the implication

$$a, b, c \geq 0,\ s > t,\ a^s \leq ba^t + c \implies a^s = O(b^{s/(s-t)} + c)$$

we obtain

$$\|x^\delta_{\alpha(\delta)} - x_0\| = O\big(\delta^\theta\big), \quad \theta = \min\Big\{1 - \eta_1, \eta_2/2\Big\}.$$

The theorem is proved. □

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