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# STABLE ITERATIVE PROCEDURES FOR A CLASS OF NONLINEAR INCREASING OPERATOR EQUATIONS IN BANACH SPACES

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Abstract. In this paper, by using weak order-Lipschitz-condition, we introduce and study a class of nonlinear equations with increasing operators, and prove the existence and uniqueness theorems of solutions for this kind of nonlinear operator equations. We also discuss the convergence and stability of perturbed iterative algorithm for solving the nonlinear operator equations, and give some applications. Our results improve and generalize the corresponding results of recent works.

## 1. Introduction and Preliminaries

Throughout this paper, we always assume that $X$ is a real Banach space with norm $\|\cdot\|$, $\theta$ is the null element of $X$ and $P \subset X$ is a cone on $X$, and the cone $P$ defines a half-order $\leq$ in $X$ by $x \leq y$ if and only if $y - x \in P$ for all $x, y \in X$. A cone $P$ in $X$ is said to be normal if, there exists a normal constant $M > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq M\|y\|$ for all $x, y \in X$. For any $u_0, v_0 \in X$, $u_0 \leq v_0$, we define the ordered interval $D = [u_0, v_0] = \{u \in X : u_0 \leq u \leq v_0\}$ (see [3, 13]).

In this paper, we need the following definitions.

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**Definition 1.1.** An operator $T : D(T) \subset X \to X$ is called to be satisfying weak order-Lipschitz-condition with operator $L$, if there exists a positive linear operator $L : X \to X$, where $\|L\| < 1$, such that

$$T(x) - T(y) \leq L(x-y), \quad \forall x, y \in D(T) \quad and \quad x \geq y.$$

If $L$ is a positive numeral function, then $T$ is called to be satisfying order-Lipschitz-condition.

**Remark 1.1.** If $T$ satisfies the order-Lipschitz-condition, then $T$ satisfies the weak order-Lipschitz-condition.

**Definition 1.2.** An operator $F : D(F) \subset X \to X$ is said to be

(i) monotone increasing, if

$$F(x) \leq F(y), \quad \forall x, y \in D(F) \quad and \quad x \leq y;$$

(ii) having inferior solution $u_0 \in D(F)$ (resp. superior solution $v_0 \in D(F)$), if

$$u_0 \leq F(u_0)(resp.\ \ F(v_0) \leq v_0).$$

Let $D \subset X$ be a subset and $T : D \to D$ be a nonlinear operator. We consider the following problem:

Find $x \in D$ such that

$$x - T(x) = 0. \tag{1.1}$$

Equation (1.1) is said to be a nonlinear operator equation.

Du [2], Guo-Lakshmikantham [3], Li [6], Li-Liang [7], Sun [8] and many other authors have studied existence and uniqueness theorems of fixed points for increasing operators by using conditions of continuity, compactness or convex-concave. Recently, Xu [10, 11] discussed existence results of fixed points for nonlinear increasing operators by using cone theory and proved the existence and uniqueness of fixed points without any compactness for operators. In 2003, by virtue of Mann iterative technique, Yu-Guo [12] proved some new theorems of solution for a class of nonlinear operator equations (1.1) without the assumption of continuity, compactness or convex-concave.

On the other hand, stability results for certain classes of nonlinear mappings have been shown in resent papers by many authors (see, for example, [1, 4, 5] and the references therein).

In this paper, by using weak order-Lipschitz-condition, we introduce and study a class of nonlinear equations with increasing operators, and prove the existence and uniqueness theorems of solutions for this nonlinear operator equations. We also discuss the convergence and stability of perturbed iterative algorithm for solving the nonlinear operator equations. As applications, we work out a two point boundary value problem of the ordinary differential equation and a Hammerstein integral equation on $R^N$ to illustrate our results.

## 2. Main Results

For our results, we need the following concept and lemma.

**Definition 2.1.** Let $S$ be a selfmap of $X$ and $x_{n+1} = h(S, x_n)(n \geq 0)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in $X$. Suppose that $\{x \in X | Sx = x\} \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point $x^*$ of $S$. Let $\{u_n\} \subset X$ and $\epsilon_n = \|u_{n+1} - h(S, u_n)\|$. If $\lim \epsilon_n = 0$ implies that $u_n \to x^*$, then the iteration procedure defined by $x_{n+1} = h(S, x_n)$ is said to be $S$-stable or stable with respect to $S$.

**Lemma 2.1.** ([9]) Let $\{\gamma_n\}$ be a nonnegative real sequence and $\{\lambda_n\}$ be a real sequence in $[0, 1]$ such that $\sum\limits_{n=0}^{\infty} \lambda_n = \infty$. If there exists a positive integer $N$ such that

$$\gamma_{n+1} \leq (1 - \lambda_n)\gamma_n + \lambda_n \sigma_n, \quad \forall n \geq N,$$

where $\sigma_n \geq 0$ for all $n \geq 0$ and $\sigma_n \to 0$ as $n \to \infty$, then $\lim\limits_{n\to\infty} \gamma_n = 0$.

**Algorithms 2.1.** *For given $x_0 \in D \subset X$, the sequence $\{x_n\}$ is defined by:*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(x_n), \quad n \geq 0,$$

*where $\{\alpha_n\}$ is a real sequence in $[0, 1)$ satisfying some conditions. Let $\{y_n\}$ be any sequence in $D$ and define $\{\varepsilon_n\}$ by*

$$\varepsilon_n = \|y_{n+1} - \{\alpha_n y_n + (1 - \alpha_n)T(y_n)\}\|, \quad n \geq 0, \tag{2.1}$$

If $\alpha_n = 0$ for all $n \geq 0$, then Algorithms 2.1 is reduced to the following algorithm.

**Algorithms 2.2.** *For given $x_0 \in D \subset X$, the sequence $\{x_n\}$ is defined by:*

$$x_{n+1} = T(x_n), \quad n \geq 0.$$

*Let $\{y_n\}$ be any sequence in $D$ and define $\{\varepsilon_n\}$ by*

$$\varepsilon_n = \|y_{n+1} - T(y_n)\|, \quad n \geq 0,$$

**Remark 2.1.** The iterative procedures $\{x_n\}$ in Algorithms 2.2 is studied by many authors (see, for example, [6, 7, 10, 11] and the references therein).

**Theorem 2.1.** *Let $u_0, v_0 \in X$, $u_0 < v_0$ and $D = [u_0, v_0]$. Let $T : D \to D$ be a monotone increasing operator with inferior solution $u_0$ and superior solution $v_0$ and satisfy weak order-Lipschitz-condition with positive linear operator $L : X \to X$. Suppose following conditions hold*:

(1) $\|L\| \leq \min\{M^{-1}, 1\}$;

(2) $\alpha_n$ *is monotone increasing and* $\alpha_n \to \alpha \in [0, 1)$.

*Then the sequence $\{x_n\}$ generated by Algorithm* 2.1 *converges strongly to the unique solution $\bar{x}$ of problem* (1.1) *and*

$$\|x_n - \bar{x}\| \leq M[\epsilon + \alpha(1-\epsilon)]^n \min\{\|x_0 - u_0\| + \frac{\|u_1 - u_0\|}{(1-\alpha)(1-\epsilon)},$$
$$\|v_0 - x_0\| + \frac{\|v_0 - v_1\|}{(1-\alpha)(1-\epsilon)}\},$$

*where $u_1 = \alpha_0 u_0 + (1-\alpha_0)T(u_0)$ and $v_1 = \alpha_0 v_0 + (1-\alpha_0)T(v_0)$. Moreover, if $\sum\limits_{n=0}^{\infty}(1-\alpha_n) = \infty$, then $\lim\limits_{n\to\infty} y_n = \bar{x}$ if and only if $\lim\limits_{n\to\infty} \varepsilon_n = 0$, where $\varepsilon_n$ is defined by* (2.1).

*Proof.* Let

$$u_{n+1} = \alpha_n u_n + (1-\alpha_n)T(u_n), \quad \forall n \geq 0, \tag{2.2}$$
$$v_{n+1} = \alpha_n v_n + (1-\alpha_n)T(v_n), \quad \forall n \geq 0. \tag{2.3}$$

Since $T : D \to D$ is a monotone increasing operator with inferior solution $u_0$ and superior solution $v_0$, it follows from (2.2) and (2.3) that

$$\begin{cases} u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq v_n \leq \cdots \leq v_1 \leq v_0, \\ u_n \leq T(u_n), \quad T(v_n) \leq v_n, \quad \forall n \geq 0. \end{cases} \tag{2.4}$$

In fact, when $n = 1$, by (2.2), (2.3) and the monotone increasing property of $T$, we have

$$\begin{aligned} u_1 - u_0 &= [\alpha_0 u_0 + (1-\alpha_0)T(u_0)] - u_0 = (1-\alpha_0)[T(u_0) - u_0] \geq 0, \\ T(u_1) - u_1 &= T(u_1) - [\alpha_0 u_0 + (1-\alpha_0)T(u_0)] \\ &= T(u_1) - T(u_0) + \alpha_0[T(u_0) - u_0] \geq 0, \\ v_1 - u_1 &= [\alpha_0 v_0 + (1-\alpha_0)T(v_0)] - [\alpha_0 u_0 + (1-\alpha_0)T(u_0)] \\ &= \alpha_0(v_0 - u_0) + (1-\alpha_0)[T(v_0) - T(u_0)] \geq 0, \\ v_0 - v_1 &= v_0 - [\alpha_0 v_0 + (1-\alpha_0)T(v_0)] = (1-\alpha_0)[v_0 - T(v_0)] \geq 0 \end{aligned}$$

and

$$\begin{aligned} v_1 - T(v_1) &= [\alpha_0 v_0 + (1-\alpha_0)T(v_0)] - T(v_1) \\ &= T(v_0) - T(v_1) + \alpha_0[v_0 - T(v_0)] \geq 0. \end{aligned}$$

Therefore, $u_0 \leq u_1 \leq v_1 \leq v_0$, i.e, (2.4) holds for $n = 1$.

Suppose now that (2.4) holds for $n = k$, i.e., $u_{k-1} \leq u_k \leq v_k \leq v_{k-1}$ and $u_k \leq T(u_k)$, $T(v_k) \leq v_k$. We shall show that it holds for $n = k+1$. In fact, by the monotonicity of $T$ and induction hypothesis, we have

$$\begin{aligned} u_{k+1} - u_k &= [\alpha_k u_k + (1-\alpha_k)T(u_k)] - u_k \\ &= (1-\alpha_k)[T(u_k) - u_k] \geq 0, \\ T(u_{k+1}) - u_k &= T(u_{k+1}) - [\alpha_k u_k + (1-\alpha_k)T(u_k)] \\ &= T(u_{k+1}) - T(u_k) + \alpha_k[T(u_k) - u_k] \geq 0, \\ v_{k+1} - u_{k+1} &= [\alpha_k v_k + (1-\alpha_k)T(v_k)] - [\alpha_k u_k + (1-\alpha_k)T(u_k)] \\ &= \alpha_k(v_k - u_k) + (1-\alpha_k)[T(v_k) - T(u_k)] \geq 0, \\ v_k - v_{k+1} &= v_k - [\alpha_k v_k + (1-\alpha_k)T(v_k)] = (1-\alpha_k)[v_k - T(v_k)] \geq 0, \\ v_{k+1} - T(v_{k+1}) &= [\alpha_k v_k + (1-\alpha_k)T(v_k)] - T(v_{k+1}) \\ &= T(v_k) - T(v_{k+1}) + \alpha_k[v_k - T(v_k)] \geq 0. \end{aligned}$$

Thus $u_k \leq u_{k+1} \leq v_{k+1} \leq v_k$ and $u_{k+1} \leq T(u_{k+1})$, $T(v_{k+1}) \leq v_{k+1}$. Therefore, (2.4) is true. Again, since $T$ satisfies weak order-Lipschitz-condition with operators $L$ and $\{\alpha_n\}$ is a monotone increasing sequence, it follows from (2.2) that

$$\begin{aligned} \theta &\leq (u_{n+1} - u_n) - (u_n - u_{n-1}) \\ &= (1-\alpha_n)[T(u_n) - u_n] - (1-\alpha_{n-1})[T(u_{n-1}) - u_{n-1}] \\ &\leq (1-\alpha_{n-1})\{[T(u_n) - T(u_{n-1})] - (u_n - u_{n-1})\} \\ &\leq (1-\alpha_{n-1})(L-I)(u_n - u_{n-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} \theta &\leq (u_{n+1} - u_n) \\ &\leq [(1-\alpha_{n-1})L + \alpha_{n-1}I](u_n - u_{n-1}) \\ &\leq [(1-\alpha_{n-1})L + \alpha_{n-1}I][(1-\alpha_{n-2})L + \alpha_{n-2}I](u_{n-1} - u_{n-2}) \\ &= Q_{n-1}Q_{n-2}(u_{n-1} - u_{n-2}) \qquad (2.5) \\ &\leq \cdots \\ &\leq (u_1 - u_0)\prod_{i=0}^{n-1} Q_i, \end{aligned}$$

where $Q_i = (1-\alpha_i)L + \alpha_i I$ for $i = 0, 1, 2, \cdots$. Since $\alpha_n \in [0, 1)$ is monotone increasing and $\alpha_n \to \alpha \in [0, 1)$, for any $\epsilon \in (\|L\|, 1)$ and $i = 0, 1, 2, \cdots$, we get

$$\|Q_i\| \leq (1-\alpha_i)\|L\| + \alpha_i \leq \epsilon(1-\alpha_i) + \alpha_i \leq \sigma,$$

where $\sigma = \epsilon + \alpha(1-\epsilon)$ and $0 < \sigma < 1$. It follows from (2.5) and the normality of $P$ that

$$\|u_{n+1} - u_n\| \leq M\|u_1 - u_0\|\sigma^n. \tag{2.6}$$

For any $n, m \geq 1$, (2.6) implies

$$\begin{aligned} \|u_{n+m} - u_n\| &\leq \sum_{i=1}^{m} \|u_{n+i} - u_{n+i-1}\| \leq \sum_{i=1}^{m} M\|u_1 - u_0\|\sigma^{n+i-1} \\ &\leq \frac{M\sigma^n(1-\sigma^m)}{1-\sigma}\|u_1 - u_0\|. \end{aligned} \tag{2.7}$$

It follows from (2.7) that $\{u_n\}$ is a Cauchy sequence. The completeness of $X$ and $u_n \in D$ imply that there exists $x_* \in D$ such that $u_n \to x_*$ as $n \to +\infty$. Similarly, we can obtain that $\{T(u_n)\}$ is also Cauchy sequence and so there exists $w_* \in D$ such that $T(u_n) \to w_*$ $(n \to \infty)$.

Next, we prove that $T(x_*) = x_* = w_*$. Indeed, letting $n \to \infty$ in (2.2), we have

$$x_* = \alpha x_* + (1-\alpha)w_*$$

and so $x_* = w_*$. Since $T$ is monotone increasing and $u_n \leq x_*$, $T(u_n) \leq T(x_*)$. It follows from $T(u_n) \to w_*$ $(n \to \infty)$ that $w_* \leq T(x_*)$. On the other hand, $T(u_n) \leq T(x_*)$ implies

$$\theta \leq T(x_*) - T(u_n) \leq L(x_* - u_n)$$

and so

$$\|T(x_*) - T(u_n)\| \leq M\|L\|\|x_* - u_n\|,$$

i.e., $T(u_n) \to T(x_*)$ as $n \to \infty$. Since $\{T(u_n)\}$ is a monotone increasing sequence, $T(u_n) \leq w_*$ and so $T(x_*) \leq w_*$. It follows from $w_* \leq T(x_*)$ that $w_* = T(x_*) = x_*$.

By the same method as above, we can know that $\{v_n\}$ and $\{T(v_n)\}$ are also Cauchy sequences, and there exists $x^* \in D$ such that $v_n \to x^*$ as $n \to +\infty$ and $T(x^*) = x^*$. Since $x_0 \in [u_0, v_0]$, if $u_k \leq x_k \leq v_k$, then

$$T(u_k) \leq T(x_k) \leq T(v_k)$$

and

$$\alpha_k u_k + (1-\alpha_k)T(u_k) + (1-\alpha_k)\omega_k \leq \alpha_k x_k + (1-\alpha_k)T(x_k) + (1-\alpha_k)\omega_k$$
$$\leq \alpha_k v_k + (1-\alpha_k)T(v_k) + (1-\alpha_k)\omega_k,$$

i.e.,

$$u_{k+1} \leq x_{k+1} \leq v_{k+1}.$$

By induction, we know that $u_n \leq x_n \leq v_n$ for all $n \geq 0$. If $\bar{x}$ is a fixed point of $T$ in $D$, then $x_* \leq \bar{x} \leq x^*$ and

$$\theta \leq x^* - x_* = T(x^*) - T(x_*) \leq L(x^* - x_*),$$

and so

$$\|x^* - x_*\| \leq M\|L\|\|x^* - x_*\|. \tag{2.8}$$

It follows from (2.8) and $\|L\| \leq M^{-1}$ that $x_* = x^*$ and so $T$ has a unique fixed point $\bar{x}$ in $D$, and $x_n \to \bar{x}$. In fact, Since $u_n \leq x_n \leq v_n$ for all $n \geq 0$,

$$\theta \leq x_n - u_n = \alpha_{n-1}(x_{n-1} - u_{n-1}) + (1-\alpha_{n-1})[T(x_{n-1}) - T(u_{n-1})].$$

By the proof of (2.6), we have

$$\|x_n - u_n\| \leq M\sigma^n\|x_0 - u_0\|. \tag{2.9}$$

Letting $m \to \infty$ in (2.7), we have

$$\|\bar{x} - u_n\| = \|x_* - u_n\| \leq \frac{M\sigma^n\|u_1 - u_0\|}{1-\sigma}. \tag{2.10}$$

Combining (2.9) and (2.10), we get

$$\begin{aligned}\|x_n - \bar{x}\| &\leq \|x_n - u_n\| + \|u_n - \bar{x}\| \\ &\leq M\sigma^n(\|x_0 - u_0\| + \frac{\|u_1 - u_0\|}{1-\sigma}) \\ &= M[\epsilon + \alpha(1-\epsilon)]^n[\|x_0 - u_0\| + \frac{\|u_1 - u_0\|}{(1-\alpha)(1-\epsilon)}],\end{aligned} \tag{2.11}$$

and so $x_n \to \bar{x}$. Similarly, we have

$$\begin{aligned}\|x_n - \bar{x}\| &\leq \|v_n - x_n\| + \|v_n - \bar{x}\| \\ &\leq M[\epsilon + \alpha(1-\epsilon)]^n[\|v_0 - x_0\| + \frac{\|v_0 - v_1\|}{(1-\alpha)(1-\epsilon)}].\end{aligned} \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$\|x_n - \bar{x}\| \leq M[\epsilon + \alpha(1-\epsilon)]^n \min\{\|x_0 - u_0\| + \frac{\|u_1 - u_0\|}{(1-\alpha)(1-\epsilon)}, \\ \|v_0 - x_0\| + \frac{\|v_0 - v_1\|}{(1-\alpha)(1-\epsilon)}\}.$$

Now, we prove $y_n \to \bar{x}$ if and only if $\varepsilon_n \to 0$. By (2.1), we obtain

$$\begin{aligned} \|y_{n+1} - \bar{x}\| &\leq \|y_{n+1} - \{\alpha_n y_n + (1-\alpha_n)T(y_n)\}\| \\ &\quad + \|\alpha_n y_n + (1-\alpha_n)T(y_n) - \bar{x}\| \\ &= \|\alpha_n y_n + (1-\alpha_n)T(y_n) - \bar{x}\| + \varepsilon_n. \end{aligned} \tag{2.13}$$

Without losing generality, let $y_n \geq \bar{x}$ for all $n \geq 0$. Since $T$ satisfies weak order-Lipschitz condition, we have

$$\begin{aligned} \|\alpha_n y_n + (1-\alpha_n)T(y_n) - \bar{x}\| &\leq \alpha_n \|y_n - \bar{x}\| + (1-\alpha_n)\|T(y_n) - T(\bar{x})\| \\ &\leq [\alpha_n + (1-\alpha_n)\|L\|]\|y_n - \bar{x}\| \\ &= [1 - (1-\alpha_n)(1-\|L\|)]\|y_n - \bar{x}\|. \end{aligned} \tag{2.14}$$

Since $0 \leq \alpha_n \leq \alpha$, it follows from (2.13) and (2.14) that

$$\begin{aligned} \|y_{n+1} - \bar{x}\| &\leq [1 - (1-\alpha_n)(1-\|L\|)]\|y_n - \bar{x}\| \\ &\quad + (1-\alpha_n)(1-\|L\|)[\frac{1}{1-\|L\|}\frac{\varepsilon_n}{1-\alpha}]. \end{aligned} \tag{2.15}$$

If $\lim_{n\to\infty} \varepsilon_n = 0$, then $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$. Therefore, (2.15) and Lemma 2.1 imply $\lim_{n\to\infty} y_n = x$.

Conversely, if $\lim_{n\to\infty} y_n = x$, then from (2.1) and (2.14), we get

$$\begin{aligned} \varepsilon_n &\leq \|y_{n+1} - \bar{x}\| + \|\alpha_n y_n + (1-\alpha_n)T(y_n) - \bar{x}\| \\ &\leq \|y_{n+1} - \bar{x}\| + [1 - (1-\alpha_n)(1-\|L\|)]\|y_n - \bar{x}\| \to 0 \quad (n \to \infty). \end{aligned}$$

This completes the proof. $\square$

**Theorem 2.2.** *Let $u_0, v_0 \in X$, $u_0 < v_0$ and $D = [u_0, v_0]$. Let $T$ be the same as in Theorem* 2.1. *If $\|L\| \leq \min\{M^{-1}, 1\}$, then the sequence $\{x_n\}$ generated by Algorithm* 2.2 *converges strongly to the unique solution $\bar{x}$ of problem* (1.1) *and for any $\epsilon \in (\|L\|, 1)$, we have*

$$\|x_n - \bar{x}\| \leq M\epsilon^n \min\{\|x_0 - u_0\| + \frac{\|u_1 - u_0\|}{1-\epsilon}, \\ \|v_0 - x_0\| + \frac{\|v_0 - v_1\|}{1-\epsilon}\},$$

*where $u_1 = T(u_0)$ and $v_1 = T(v_0)$. Moreover, $\lim_{n\to\infty} y_n = \bar{x}$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$, where $\varepsilon_n$ is defined by Algorithm* 2.2.

**Remark 2.2.** In Theorems 2.1 and 2.2, we have not required any compactness, continuity, strongly increasing property and convex-concave condition. Our results improve and generalize many known corresponding results, see, for example, [6, 7, 10-12] and the references therein.

## 3. Applications

In this section, we will consider some examples by using our results in section 2.

**Definition 3.1.** A operator $f:\ R\times X \to X$ is said to be monotone increasing with respect to the second argument if

$$f(t, x_1) \leq f(t, x_2),$$

where $0 \leq t \leq 1$, $\theta \leq x_1 \leq x_2$.

Let $f : [0,1] \times [0, +\infty) \to (0, +\infty)$ be a continuous function and $f(t, 0) = 0$ for all $t \in [0, 1]$. We denote by $C[0, 1]$ the space of continuous functions on $[0, 1]$ and $C^2[0, 1]$ the class of functions having continuous second derivative on $[0, 1]$. Assume that $f$ satisfies following conditions :

- (i) $f(t, x)$ is monotone increasing with respect to $x$;
- (ii) $f(t, x) > 0$ for all $0 < t < 1$, $x > 0$;
- (iii) $\frac{f(t,x)}{x}$ converges uniformly to 0 with respect to $t \in [0, 1]$ as $x \to \infty$;
- (iv) there exists a constant function $L :\ [0, +\infty) \to [0, +\infty)$ with $0 < L < 1$ such that for any $x(t), y(t) \in C[0, 1]$ with $x(t) \geq y(t)$,

$$\lambda \int_0^1 g(t, s)[f(s, x(s)) - f(s, y(s))]ds \leq L(x(t) - y(t)),$$

where $\lambda$ is a parameter and

$$g(t,s) = \begin{cases} t(1-s), t \le s, \\ s(1-t), t > s, \end{cases}$$

is a Green function corresponding to the following two point boundary value problem of the ordinary differential equation:

$$\begin{cases} \frac{d^2x}{dt^2} = \lambda f(t,x), \\ x(0) = x(1) = 0. \end{cases} \tag{3.1}$$

**Theorems 3.1.** *Let $f$ satisfies the conditions (i)-(iv). Then*

(1) *for any $M > 0$, there exists a real number $h$ such that for all $\lambda \ge h$, problem (3.1) has a unique continuous positive solution $x_\lambda(t) \in C^2[0,1]$ satisfying $x_\lambda(t) \ge Mt(1-t)$ for all $0 \le t \le 1$;*

(2) *for initial value $x_0(t) = Mt(1-t)$, if the iterative sequence $\{x_n(t)\}$ is defined as follows :*

$$x_{n+1}(t) = \alpha_n x_n(t) + (1-\alpha_n)\lambda \int_0^1 g(t,s) f(s, x_n(s)) ds, \quad n \ge 0,$$

*where $\alpha_n \in [0,1)$ is monotone increasing and $\alpha_n \to \alpha \in [0,1)$, $\sigma = \epsilon + \alpha(1-\epsilon)$, $M \int_0^1 g(t,s)ds < \epsilon < 1$, then $\{x_n\}$ converges uniformly to $x_\lambda(t)$ and*

$$\begin{aligned} &\sup_{t\in[0,1]} \|x_n(t) - x^*(t)\| \\ &\le M[\epsilon + \alpha(1-\epsilon)]^n \min\{\sup_{t\in[0,1]} \|x_0(t) - u_0\| \\ &+ \frac{1-\alpha_0}{(1-\alpha)(1-\epsilon)} \|T(u_0) - u_0\|, \\ &\quad \sup_{t\in[0,1]} \|c - x_0\| + \frac{1-\alpha_0}{(1-\alpha)(1-\epsilon)} \|c - T(c)\|\}, \end{aligned}$$

*where $T(u) = \int_0^1 g(t,s) f(s,u) ds$, $u_0 = \inf_{t\in[0,1]} Mt(1-t)$ and $v_0 = c > M$;*

(3) *if* $\sum_{n=0}^{\infty}(1-\alpha_n)=\infty$, *then* $\lim_{n\to\infty} y_n(t)=x_\lambda(t)$ *if and only if* $\lim_{n\to\infty}\varepsilon_n(t)=0$ *for all* $t\in[0,1]$, *where* $\{y_n(t)\}$ *be any sequence in* $[u_0,v_0]$ *and* $\varepsilon_n(t)$ *is defined by*

$$\varepsilon_n(t)=\|y_{n+1}(t)-\{\alpha_n y_n(t)+(1-\alpha_n)T(y_n(t))\}\|,\quad n\ge 0.$$

*Proof.* Obviously, $x_\lambda(t)\in C^2[0,1]$ is a solution of the problem (3.1) if and only if it is a solution of the following integral equation on $C[0,1]$:

$$x(t)=\lambda\int_0^1 g(t,s)f(s,x(s))ds.$$

Let

$$T(x(t))=\lambda\int_0^1 g(t,s)f(s,x(s))ds,$$

and $X=C[0,1]$, $P=\{x(t)\in C[0,1]|\,x(t)\ge 0,\forall 0\le t\le 1\}$. It is easy to know that $P$ is a normal cone on $X$ and the operator $T:P\to X$ is continuous. Let $u_0(t)=Mt(1-t)$. Then from [3], for any $\lambda\ge h$ we have $\lambda T(u_0(t))\ge u_0(t)$ for all $t\in[0,1]$. Since $\frac{f(t,x)}{x}$ converges uniformly to 0 with respect to $t\in[0,1]$ as $x\to\infty$, there exists a constant $c>M$ such that $\frac{f(t,c)}{c}\le\frac{8}{\lambda}$ for all $t\in[0,1]$. Let $v_0(t)\equiv c$. Then for every $t\in[0,1]$, $u_0(t)<v_0(t)$ and $\lambda T(v_0(t))\le v_0(t)$. It follows from (iv) that there exists a constant function $L:[0,+\infty)\to[0,+\infty)$ with $0<L<1$ such that for any $x(t),y(t)\in[u_0,v_0]=\{z(t)\in C[0,1]|\,u_0(t)\le z(t)\le v_0(t),\forall t\in[0,1]\}$ with $x(t)\ge y(t)$, we have

$$\lambda T(x(t))-\lambda T(y(t))\le L(x(t)-y(t)).$$

Since $f(t,x)$ is monotone increasing with respect to $x$, operator $\lambda T:[u_0,v_0]\to X$ is monotone increasing. Therefore, from Theorem 2.1, the proof is completed.

□

**Theorems 3.2.** *Let* $f:R^N\times[0,+\infty)\to R$ *be a increasing function, where* $R^N$ *denotes n-dimension Euclidean space and* $R$ *is the set of all real numbers. Assume that*

(i) $k(t,s)$ *is a nonnegative measurable function on* $R^N\times R^N$ *and*

$$\lim_{t\to t_0}\int_{R^N}|k(t,s)-k(t_0,s)|ds=0$$

*for all* $t_0 \in R^N$;

(ii) *there exist* $m, M \in R$ *such that* $0 < m < \int_{R^N} k(t,s)ds \leq M$ *for all* $t \in R^N$;

(iii) $f(t,u)$ *satisfies Caratheodary condition, i.e., for any* $u \in [0,\infty)$, $f(\cdot,u)$ *is a measurable function on* $R^N$ *and* $f(t,\cdot)$ *is a continuous function on* $[0,\infty)$ *for all* $t \in R^N$;

(iv) *there exist* $l, q > 0$ *such that* $f(t,l) \geq \frac{l}{m}$ *and* $f(t,q) \leq \frac{q}{m}$ *for all* $t \in R^N$;

(v) *there exists a constant* $d \in (0, \frac{1}{M})$ *such that*

$$\begin{aligned}
&f(t,x(t)) - f(t,y(t)) \leq d(x(t) - y(t)),\\
&\forall t \in R^N, \quad x(t),\, y(t) \in C_B(R^N) \quad \textit{and} \quad x(t) \geq y(t),
\end{aligned}$$

*where* $C_B(R^N)$ *is the family of continuous and bounded function.*

*Then*

(1) *the following Hammerstein integral equation on* $R^N$:

$$x(t) = \int_{R^N} k(t,s)f(s,x(s))ds$$

*has a unique continuous bounded positive solution* $x^*(t) \in R$ *for all* $t \in R^N$;

(2) *for any initial value* $x_0(t) \in R$, *the iterative sequence* $\{x_n(t)\}$ *defined by*

$$x_{n+1}(t) = \alpha_n x_n(t) + (1-\alpha_n)\int_{R^N} k(t,s)f(s,x_n(s))ds, \quad n \geq 0,$$

*converges uniformly to* $x^*(t)$, *where* $\alpha_n \in [0,1)$ *is monotone increasing and* $\alpha_n \to \alpha \in [0,1)$. *And for any* $M\int_{R^N} k(t,s)ds < \epsilon < 1$, *we have*

$$\begin{aligned}
&\sup_{t\in R^N} \|x_n(t) - x^*(t)\|\\
&\leq M[\epsilon + \alpha(1-\epsilon)]^n \min\{\sup_{t\in R^N} \|x_0(t) - l\|\\
&\quad + \frac{1-\alpha_0}{(1-\alpha)(1-\epsilon)}\|T(l) - l\|,\\
&\quad \sup_{t\in R^N} \|q - x_0\| + \frac{1-\alpha_0}{(1-\alpha)(1-\epsilon)}\|q - T(q)\|\},
\end{aligned}$$

*where* $T(u) = \int_{R^N} k(t,s)f(s,u)ds$;

(3) *if* $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$, *then* $\lim_{n\to\infty} y_n(t) = x^*(t)$ *if and only if* $\lim_{n\to\infty} \varepsilon_n(t) = 0$ *for all* $t \in R^N$, *where* $\{y_n(t)\}$ *be any sequence in* $[l,q]$ *and* $\varepsilon_n(t)$ *is defined by*

$$\varepsilon_n(t) = \|y_{n+1}(t) - \{\alpha_n y_n(t) + (1-\alpha_n)T(y_n(t))\}\|, \quad n \geq 0.$$

*Proof.*. Let $\|x\|_{C_B} = \sup_{t\in R^N} \|x(t)\|$ be a norm on $C_B(R^N)$. We denote by $C_{B^+}(R^N)$ the class of nonnegative functions on $C_B(R^N)$. Then $C_{B^+}(R^N)$ is a normal cone with normal constant $M$ on $C_B(R^N)$. Letting

$$X = C_B(R^N), \quad P = C_{B^+}(R^N),$$

$$u_0 = u_0(t) \equiv l, \quad v_0 = v_0(t) \equiv q, \quad D = \{x \in C_B(R^N) |\, u_0 \leq x \leq v_0\}$$

and

$$T(x(t)) = \int_{R^N} k(t,s)f(s,x(s))ds.$$

Then $T : D \to D$ is monotone increasing and for any $x, y \in D$, $x \geq y$, i.e., $x(t) \geq y(t)$ for all $t \in R^N$, we have

$$\begin{aligned} T(x(t)) - T(y(t)) &= \int_{R^N} k(t,s)f(s,x(s))ds - \int_{R^N} k(t,s)f(s,y(t))ds \\ &\leq \int_{R^N} k(t,s)d(x(s) - y(t))ds. \end{aligned}$$

Now we prove that $\|L\| \leq 1$, where $L(w) = \int_{R^N} k(t,s)dw(s)ds$ is a linear operator on $C_B(R^N)$. In fact, for any $t \in R^N$, $w \in C_B(R^N)$, we have

$$\|L(w)\| \leq \int_{R^N} k(t,s)d\|w(s)\|ds \leq dM\|w\|_{C_B}$$

and so $\|L\| \leq dM < 1$. It follows from Theorem 2.1 that the conclusions of Theorem 3.2 are obtained.

□

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