Nonlinear Funct. Anal. & Appl., Vol. 10, No. 3 (2005), pp. 415–425

SOME EXISTENCE RESULTS FOR GENERALIZED VECTOR QUASI-VARIATIONAL INEQUALITIES

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Abstract. Some existence results for vector quasi-variational inequalities involving multi-valued mappings in topological vector spaces are derived both under compact and noncompact assumptions by employing the one person game theorems. The results of this paper generalize and unify the corresponding results of several authors and can be considered as a significant extension of the previously known results.

1. Introduction

Let K be a nonempty set and $f : K \times K \to R$ be a bifunction. The equilibrium problem [2] is defined to be the problem of finding a point $x \in K$ such that $f(x, y) \geq 0$ for each $y \in K$. It is well known that equilibrium problems are closely related to the game theory, economics and finance, mechanics and physics and operation research, and are unified mathematical model of several problems, for instance, optimization problems, variational inequalities, complementarity problems, fixed point problems and saddle point problems. Generally the set involved in the formulation of the variational inequality and equilibrium, problems does not depend on the solution of the problem. If the set does depend on the solution, then a problem in this class is called quasi-variational inequality and quasi-equilibrium problem respectively. Lin and Park [19] and Ding [6, 7] introduced and studied the quasi-equilibrium problems in G-convex spaces and general topological spaces using fixed point approach.

Received March 19, 2004. Revised November 16, 2004.

2000 Mathematics Subject Classification: 90C29, 49J35, 49J40, 47J20.

Key words and phrases: Quasi-equilibrium problems, quasi-variational inequalities, one person game theorems, upper semicontinuous multi-valued mapping.

The first author was supported by the University Grants Commision, India, Project No. DAA/2001/4562.

Recently inspired by the concept of vector variational inequality problem and its various extensions (see [11] and the references cited therein), quasi variational inequalities and quasi-equilibrium problems has been extended to vector-valued functions by many authors (see for example [6, 9, 11, 13-17]). In this paper, we introduce a new class of *vector quasi-variational inequality problem* involving multi-valued mappings, which extends and generalizes the known equilibrium problems, and the corresponding results in [1, 2, 6, 7, 8, 11, 12-22]. We derive existence results for solution of the vector quasi-variational inequality problem both under compact and noncompact settings.

2. Formulations and Preliminaries

Before the formal discussion, we introduce some notions and definitions. Let X be a vector space and $K \subset X$. We shall denote by $co(K)$, the *convex hull* of K. If K is a subset of a topological space X, the *interior* of K in X is denoted by $int_X(K)$ and *closure* of K in X is denoted by $cl_X(K)$ or simply $int(K)$, and $cl(K)$ if there is no ambiguity, respectively. Let X and Y be two sets, we shall denote by 2^X the family of all subsets of X and if $F, G : X \to 2^Y$ be multi-valued mappings then the mapping $F \cap G : X \to 2^Y$ is defined by $(F \cap G)(x) = F(x) \cap G(x)$ for each $x \in X$.

Let X and Y be topological vector spaces and $T : X \to 2^Y$ be a multi-valued mapping, the *graph* of T denoted by $G(T)$, is the set $\{(x, y) \in X \times Y : x \in X, y \in T(x)\}$ and the multi-valued mapping $\overline{T} : X \to 2^Y$ is defined by $\overline{T}(x) = \{y \in Y : (x, y) \in cl_{X\times Y}G(T)\}$. The set $cl_{X\times Y}G(T)$ is called *adherence* of the graph of T. The multi-valued mapping $clT : X \to 2^Y$ is defined by $(clT)(x) = clT(x)$ for each $x \in X$. It can be seen easily that $clT(x) \subset \overline{T}(x)$ for each $x \in X$. The *inverse* of T denoted by T^{-1} is a multi-valued mapping from $R(T)$, range of T, to X defined by $x \in T^{-1}(y)$ if and only if $y \in T(x)$. Also T is said to be *upper semicontinuous* on X if for each $x \in X$ and each open set U in Y containing $T(x)$, there exists an open neighborhood V of x in X such that $T(y) \subseteq U$, for each $y \in V$. We denote by $L(X, Y)$, the space of all continuous linear operators from X to Y and by $\langle u, x\rangle$ the evaluation of $u \in L(X, Y)$ at $x \in X$. Let K be a non-empty convex subset of X and $C : K \to 2^Y$ be a multi-valued mapping such that for each $x \in K, C(x)$ is a closed convex cone with $intC(x) \neq \emptyset$, where $intC(x)$ denotes the interior of $C(x)$. It is clear that the cone $C(x)$ for each $x \in K$ can define on Y a partial order $\preceq_{C_x}$ by $y \preceq_{C_x} z$ if and only if $z - y \in C(x)$. We shall write $y \prec_{C_x} z$ if $z-y \in intC(x)$ in the case $intC(x) \neq \emptyset$. The multi-valued mapping $T : K \to 2^Y$ is said to be $C_x - convex$ if for each $x, y \in K$ and $\lambda \in [0, 1]$, $T(\lambda y + (1 - \lambda)x) \preceq_{C_x} \lambda T(y) + (1 - \lambda)T(x)$.

Let $T : K \to 2^{L(X,Y)}$ be a multi-valued mapping. For a given continuous multi-valued mapping $A : K \to 2^K$ and a vector-valued bifunction $f : K \times K \to Y$, we consider the following *vector quasi-variational inequality problem* (for short, VQVIP): Find $x^* \in K$ such that for all $y \in A(x^*)$ there is $t^* \in T(x^*)$ such that

$$x^* \in cl_K A(x^*) \text{ and } \langle t^*, y - x^* \rangle + f(x^*, y) \notin -int_Y C(x^*). \tag{2.1}$$

If $T \equiv 0$, then the (VQVIP) reduces to the problem of finding $x^* \in K$ such that for all $y \in A(x^*)$

$$x^* \in cl_K A(x^*) \text{ and } f(x^*, y) \notin -int_Y C(x^*). \tag{2.2}$$

It is called *vector quasi-equilibrium problem* (for short, VQEP) considered by Khaliq and Krishan [16].

If $f(x,y) \equiv 0, A(x) \equiv K$ for all $x, y \in K$, (VQVIP) becomes *generalized vector variational inequality problem* (for short, GVVIP) of finding $x^* \in K$ such that for all $y \in K$ there is $t^* \in T(x^*)$ such that

$$\langle t^*, y - x^* \rangle \notin -int_Y C(x^*). \tag{2.3}$$

This problem was introduced and studied by Lin et al.[20] and Konnov and Yao [18].

If T is single valued mapping, $A(x) = K, C(x) = P$ and $f(x,y) = \langle S(x), y - x \rangle$ for all $x, y \in K$, (VQVIP) reduces to the problem of finding $x^* \in K$ such that

$$\langle S(x^*) + T(x^*), y - x^* \rangle \notin -int_Y P, \text{ for all } y \in K, \tag{2.4}$$

which is known as *strongly nonlinear vector variational inequality problem* (for short, SNVVIP) studied by Ansari [1].

The vector variational inequalities are very useful from the application point of view in optimization, optimal control, economic equilibrium and free boundary value problems and have been shown to be a useful tool in the geometrical features of optimization.

The following one person game theorems will be used to establish the main results of this paper.

Theorem 2.1. *Let $\Gamma = (K; A, P)$ be a 1-person game such that*

(i) *K is a nonempty compact convex subset of a Hausdorff topological vector space,*

(ii) *$A, cl_X A : K \to 2^K$ be multi-valued mappings such that for each $x \in K, A(x)$ is nonempty convex set in K, for each $y \in K, A^{-1}(y)$ is open set in K and $cl_X A$ is upper semicontinuous,*

(iii) *$P : K \to 2^K$ be a multi-valued mapping such that for each $x \in K, x \notin coP(x)$ and for each $y \in K, P^{-1}(y)$ is open set in K.*

Then there exists $x^* \in K$ such that $x^* \in cl_K A(x^*)$ and $A(x^*) \bigcap P(x^*) = \emptyset$.

Theorem 2.2. *Let $\Gamma = (K; A, P)$ be a 1-person game such that*

(i) *K is a nonempty convex subset of a locally convex Hausdorff topological vector space and D be a nonempty compact subset of K,*

(ii) *$A : K \to 2^D$ and $cl_X A : K \to 2^K$ be multi-valued mappings such that for each $x \in K$, $A(x)$ is nonempty convex set, for each $y \in D, A^{-1}(y)$ is open set in K and $cl_X A$ is upper semicontinuous,*

(iii) *$P : K \to 2^D$ be a multi-valued mapping such that for each $x \in K, x \notin coP(x)$ and for each $y \in D, P^{-1}(y)$ is open in K.*

Then there exists $x^* \in K$ such that $x^* \in cl_K A(x^*)$ and $A(x^*) \bigcap P(x^*) = \emptyset$.

Remark 2.1. Theorem 2.1 is a special case of [9, Theorem 2] and Theorem 2.2 is a special case of [10, Theorem 2].

3. Existence Results

In this section we establish some existence results under compact and non-compact assumptions. We need the following:

Lemma 3.1 [8]. *Let X and Y be topological vector spaces and let $L(X, Y)$ be equipped with the uniform convergence topology δ. Then the bilinear form $\langle ., . \rangle : L(X, Y) \times X \to Y$ is continuous on $(L(X, Y), \delta) \times X$.*

Now we are ready to establish the main result of this paper on the existence of a solution of (VQVIP).

Theorem 3.1. *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X and Y be an ordered Hausdorff topological vector space. Let $f : K \times K \to Y$ be a vector-valued bifunction and $T : K \to 2^{L(X,Y)}$ a multi-valued mapping with compact values. Let $C : K \to 2^Y$ and $A : K \to 2^K$ be the multi-valued mappings. Assume that*

(i) *for each $x \in K$, $f(x, x) = 0$,*

(ii) *f is continuous in the first argument and $C_x -$ convex in the second argument,*

(iii) *the mapping* $W : K \to 2^Y$ *defined by* $W(x) = Y \setminus (-int_Y C(x))$ *for each* $x \in K$, *has a closed graph in* $K \times Y$,

(iv) *for each* $x \in K$, $C(x)$ *is closed, convex and pointed cone in* Y *such that* $int_Y C(x)$ *is nonempty,*

(v) *for each* $x \in K$, $A(x)$ *is nonempty convex and for each* $y \in K$, $A^{-1}(y)$ *is open in* K. *Also* $cl_K A : K \to 2^K$ *is upper semicontinuous.*

Then there exists $x^* \in K$ *such that for all* $y \in A(x^*)$ *there is* $t^* \in T(x^*)$ *such that*

$$x^* \in cl_K A(x^*) \text{ and } \langle t^*, y - x^*\rangle + f(x^*, y) \notin -int_Y C(x^*).$$

Proof. Define a multi-valued mapping $P : K \to 2^K$ as

$$P(x) = \{y \in K : \langle T(x), y - x\rangle + f(x, y) \subseteq -int_Y C(x)\}, \quad \text{for all} \quad x \in K.$$

We show first that $x \notin coP(x)$, for each $x \in K$. Suppose that $x \in coP(x)$, for some $x \in K$. Then there exists $x_o \in K$ such that $x_o \in coP(x_o)$. This implies that x_o can be expressed as

$$x_o = \sum_{i \in I} \lambda_i y_i, \quad \text{with} \quad \lambda_i \geq 0, \ \sum_{i \in I} \lambda_i = 1, \ i = 1, \cdots, n,$$

where $\{y_i : i \in N\}$ be a finite subset of K, $I \subset N$ be arbitrary nonempty subset where N denotes the set of natural numbers. This follows

$$\langle T(x_o), y_i - x_o\rangle + f(x_o, y_i) \subseteq -int_Y C(x_o) \quad \text{for all} \quad i = 1, \cdots, n.$$

Therefore for each $t \in T(x_o)$,

$$\sum_{i \in I} \lambda_i[\langle t, y_i - x_o\rangle + f(x_o, y_i)] \in -int_Y C(x_o). \tag{3.1}$$

By assumptions (i) and (ii) we have

$$0 = \langle t, x_o - x_o\rangle + f(x_o, x_o) \preceq_{C_{x_o}} \sum_{i \in I} \lambda_i[\langle t, y_i - x_o\rangle + f(x_o, y_i)].$$

Hence

$$\sum_{i \in I} \lambda_i[\langle t, y_i - x_o\rangle + f(x_o, y_i)] \in C(x_o). \tag{3.2}$$

From (3.1) and (3.2) we have

$$\sum_{i\in I}\lambda_i[\langle t, y_i - x_o\rangle + f(x_o, y_i)] \in \{-int_Y C(x_o)\} \cap C(x_o) = \emptyset,$$

which is a contradiction.

Now we show that for each $y \in K$ the set

$$\begin{aligned} P^{-1}(y) &= \{x \in K : y \in P(x)\} \\ &= \{x \in K : \langle T(x), y - x\rangle + f(x, y) \subseteq -int_Y C(x)\}, \end{aligned}$$

is open in K, which is equivalent to showing that the set

$$\begin{aligned} [P^{-1}(y)]^c &= K \setminus P^{-1}(y) \\ &= \{x \in K : \langle T(x), y - x\rangle + f(x, y) \nsubseteq -int_Y C(x)\} \\ &= \{x \in K : \exists\ t \in T(x) \quad \text{such that} \\ &\quad \langle t, y - x\rangle + f(x, y) \notin -int_Y C(x)\} \end{aligned}$$

is closed in K. For this purpose, let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in $[P^{-1}(y)]^c$ converging to $u \in K$. Then for each λ there is a $t_\lambda \in T(x_\lambda)$ such that

$$\langle t_\lambda, y - x_\lambda\rangle + f(x_\lambda, y) \in W(x_\lambda).$$

Since $T(x)$ is compact, without loss of generality we may assume that t_λ converges to some $t \in T(x)$. By (ii) f is continuous in the first argument and by Lemma 3.1 we have for each $y \in K$ and for all $t \in T(x)$, $x \to \langle t, y - x\rangle$ is continuous. Since W has a closed graph in $K \times Y$ by assumption (iii) we have

$$\langle t, y - u\rangle + f(u, y) \in W(u),$$

that is, $\langle t, y - u\rangle + f(u, y) \notin -int_Y C(u)$. Hence $u \in [P^{-1}(y)]^c$. From assumption (v), it follows that all the hypothesis of Theorem 2.1 are satisfied. Hence there exists $x^* \in K$ such that

$$x^* \in cl_K A(x^*) \text{ and } A(x^*) \bigcap P(x^*) = \emptyset.$$

Which implies that there exists $x^* \in K$ such that for all $y \in A(x^*)$ there is $t^* \in T(x^*)$ such that

$$x^* \in cl_K A(x^*) \text{ and } \langle t^*, y - x^*\rangle + f(x^*, y) \notin -int_Y C(x^*).$$

The proof is complete. $\square$

Corollary 3.1. *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X and Y be an ordered Hausdorff topological vector space. Let $T : K \to 2^{L(X,Y)}$ be a multi-valued mapping with compact values and $h : K \to Y$ be a continuous convex vector-valued function. Let $C : K \to 2^Y$ and $A : K \to 2^K$ be the multi-valued mappings. Assume that conditions (iii)-(v) of Theorem 3.1 holds. Then there exists $x^* \in K$ such that for all $y \in A(x^*)$ there is $t^* \in T(x^*)$ such that*

$$x^* \in cl_K A(x^*) \text{ and } \langle t^*, y - x^* \rangle + h(y) - h(x^*) \notin -int_Y C(x^*).$$

Proof. If we set $f(x, y) = h(y) - h(x)$, then we see that all the assumptions of Theorem 3.1 holds and hence the conclusion follows from Theorem 3.1. □

Corollary 3.2. *If in Corollary 3.1 we assume that $C(x) = R_+$ for each $x \in K$, $L(X, Y) = X^*$ and all the assumptions are satisfied then there exists $x^* \in K$ such that for all $y \in A(x^*)$ there is $t^* \in T(x^*)$ such that*

$$x^* \in cl_K A(x^*) \text{ and } \Re\langle t^*, x^* - y \rangle \le h(x^*) - h(y).$$

For the noncompact case we need the following concept of escaping sequences introduced in Border [3].

Definition 3.1. *Let X be a topological space and K a subset of X such that $K = \bigcup_{n=1}^{\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact sets in the sense that $K_n \subseteq K_{n+1}$ for all $n \in N$. A sequence $\{x_n\}_{n=1}^{\infty}$ in K is said to be escaping sequence from K (relative to $\{K_n\}_{n=1}^{\infty}$) if for each n there is an M such that $k \ge M$, $x_k \notin K_n$.*

Theorem 3.2. *Let K be a nonempty subset of a Hausdorff topological vector space X and $K = \bigcup_{n=1}^{\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty, compact and convex subsets of K. Let Y, f, T, C, W and A be the same as in Theorem 3.1 and satisfies all the conditions. In addition, suppose that for each sequence $\{x_n\}_{n=1}^{\infty}$ in K with $x_n \in K_n$, $n \in N$ which is escaping from K relative to $\{K_n\}_{n=1}^{\infty}$, there exists $m \in N$ and $y_m \in K_m \bigcap A(x_m)$ such that for each $x_m \in cl_K A(x_m)$, there is $t_m \in T(x_m)$ such that*

$$\langle t_m, y_m - x_m \rangle + f(x_m, y_m) \in -int_Y C(x_m). \tag{3.3}$$

Then there exists $x^* \in K$ such that for all $y \in A(x^*)$ there is $t^* \in T(x^*)$ such that

$$x^* \in cl_K A(x^*) \text{ and } \langle t^*, y - x^* \rangle + f(x^*, y) \notin -int_Y C(x^*).$$

Proof. Since for each $n \in N$, K_n is compact and convex set in X, Theorem 3.1 implies that for all $n \in N$, there exists $x_n \in K_n$ such that for all $z \in A(x_n)$ there is $t_n \in T(x_n)$ such that

$$x_n \in cl_K A(x_n) \text{ and } \langle t_n, z - x_n \rangle + f(x_n, z) \notin -int_Y C(x_n). \tag{3.4}$$

Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ be escaping from K relative to $\{K_n\}_{n=1}^{\infty}$. By assumption (3.3), there exists $m \in N$ and $z_m \in K_m \bigcap A(x_m)$ such that for each $x_m \in cl_K A(x_m)$, there is $t_m \in T(x_m)$ such that

$$\langle t_m, z_m - x_m \rangle + f(x_m, z_m) \in -int_Y C(x_m),$$

which contradicts (3.4). Hence $\{x_n\}_{n=1}^{\infty}$ is not an escaping sequence from K relative to $\{K_n\}_{n=1}^{\infty}$. Since T is a multi-valued mapping with compact values, thus using the arguments similar to those used in proving [14,Theorem 3.2] and [16,Theorem 2], there exists $r \in N$ and $x^* \in K_r$ such that $x_n \to x^*$ and there is $t \in T(x^*)$ such that $\langle t, y - x^* \rangle + f(x^*, y) \in W(x^*)$. Since $cl_K A : K \to 2^K$ is upper semicontinuous with compact values, hence there exists $x^* \in K$ such that for all $y \in A(x^*)$ there is $t^* \in T(x^*)$ such that

$$x^* \in cl_K A(x^*) \text{ and } \langle t^*, y - x^* \rangle + f(x^*, y) \notin -int_Y C(x^*).$$

The proof is complete. □

Theorem 3.3. *Let K be a nonempty convex subset of a locally convex Hausdorff topological vector space X and D be a nonempty compact subset of K. Let Y be an ordered Hausdorff topological vector space. Let $f : K \times K \to Y$ be a vector-valued bifunction, $T : K \to 2^{L(X,Y)}$ a multi-valued mapping with compact values and $C : K \to 2^Y$ a multi-valued mapping such that for each $x \in K$, $C(x)$ is closed, convex and pointed cone in Y with $int_Y C(x) \neq \emptyset$. Let $A, cl_K A : K \to 2^D$ be multi-valued mappings such that for each $x \in K$, $A(x)$ is nonempty convex, for each $y \in K$, $A^{-1}(y)$ is open in K and $cl_K A$ is upper semicontinuous. Suppose that conditions (i)-(iii) of Theorem 3.1 holds. Then there exists $x^* \in K$ such that for all $y \in A(x^*)$ there is $t^* \in T(x^*)$ such that*

$$x^* \in cl_K A(x^*) \text{ and } \langle t^*, y - x^* \rangle + f(x^*, y) \notin -int_Y C(x^*).$$

Proof. Define a multivalued mapping $P : K \to 2^K$ as

$$P(x) = \{y \in D : \langle T(x), y - x\rangle + f(x, y) \subseteq -int_Y C(x)\} \text{ for all } x \in K.$$

Then using the arguments similar to those used in proving Theorem 3.1, we have $x \notin coP(x)$ for each $x \in K$ and $P^{-1}(y)$ is open for each $y \in D$. Thus all the conditions of Theorem 2.2 are satisfied. Hence there exists $x^* \in K$ such that

$$x^* \in cl_K A(x^*) \text{ and } A(x^*) \cap P(x^*) = \emptyset.$$

Which implies that there exists $x^* \in K$ such that for all $y \in A(x^*)$ there is $t^* \in T(x^*)$ such that

$$x^* \in cl_K A(x^*) \text{ and } \langle t^*, y - x^*\rangle + f(x^*, y) \notin -int_Y C(x^*).$$

The proof is complete. □

Corollary 3.3. *Let K be a nonempty convex subset of a locally convex Hausdorff topological vector space X and D be a nonempty compact subset of K. Let Y be an ordered Hausdorff topological vector space. Let $T : K \to 2^{L(X,Y)}$ be a multi-valued mapping with compact values and $h : K \to Y$ be a continuous convex vector-valued function. Let $C : K \to 2^Y$ be a multi-valued mapping such that for each $x \in K$, $C(x)$ is closed, convex and pointed cone in Y with $int_Y C(x) \neq \emptyset$. Let $A, cl_K A : K \to 2^D$ be multi-valued mappings such that for each $x \in K$, $A(x)$ is nonempty convex, for each $y \in K$, $A^{-1}(y)$ is open in K and $cl_K A$ is upper semicontinuous. Suppose that conditions (i)-(iii) of Theorem 3.1 holds. Then there exists $x^* \in K$ such that for all $y \in A(x^*)$ there is $t^* \in T(x^*)$ such that*

$$x^* \in cl_K A(x^*) \text{ and } \langle t^*, y - x^*\rangle + h(y) - h(x^*) \notin -int_Y C(x^*).$$

Proof. If we set $f(x, y) = h(y) - h(x)$, then we see that all the assumptions of Theorem 3.3 holds and hence the conclusion follows from Theorem 3.3. □

Corollary 3.4. *If in Corollary 3.3 we assume that $C(x) = R_+$ for each $x \in K$, $L(X, Y) = X^*$ and all the assumptions are satisfied, then there exists $x^* \in K$ such that for all $y \in A(x^*)$ there is $t^* \in T(x^*)$ such that*

$$x^* \in cl_K A(x^*) \text{ and } \Re\langle t^*, x^* - y\rangle \leq h(x^*) - h(y).$$

Remark 3.1. The results of section 3 generalizes and improves the corresponding results in [1, 2, 14, 15, 17]. Corollary 3.3 is a noncompact generalization of Theorem 2.2 and Theorem 2.4 of Chowdhury and Tan [5]. We note that our proof of Corollary 3.3 depends on the existence theorem of one person game instead of generalized version of Ky Fan's minimax inequality. Corollary 3.4 is a noncompact generalization of Corollary 1 of Chowdhury and Tan [4].

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