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APPLICATION OF SCALING GROUP OF TRANSFORMATIONS TO VISCOELASTIC SECOND-GRADE FLUID FLOW

T. Tapanidis, Gr. Tsagas and H. P. Mazumdar

Abstract. A scaling group of transformations is applied to the steady two-dimensional viscoelastic second grade fluid flow over a stretching sheet. It is shown that the system remain invariant due to some relations among the parameters of the transformations. Two absolute invariants are then found out and utilized to derive a fourth-order standard differential equation corresponding to the momentum equation. This fourth-order differential equation admits an exact solution to the problem.

1. Introduction

In many applied flow problems, the partial differential equations governing the motion of the fluid are nonlinear and hence can not be solved easily. A popular goal is to obtain similarity solution wherever possible by employing transformations that reduce the system of partial differential equations to a system of ordinary differential equations. A systematic method namely, the group-theoretic method has been developed from Sophus Lie's idea of continuous group of transformations. Ames [1], Birkhoff [3], Bluman-Cole [4], Hansen [5], Ibragimov [6], Olver [7], Seshadri-Na [12] and Stephani [14] have discussed application of groups and symmetries to partial differential equations arising from physical problems along with the boundary conditions. In the present analysis, we apply a special form of Lie's group transformations namely, the scaling group of transformations to the problem of flow of viscoelastic second grade fluid over a stretching sheet.

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2. Formulation of the problem

Sakiadis [11] first studied the boundary layer flow of a viscous fluid due to the motion of an inextensible plane sheet in its own plane. Beard-Walters [2], Rajgopal et al. [9, 10], Siddappa-Abel [13] and many others have investigated the boundary layer flow of viscoelastic fluid past a stretching sheet. The results of this type of investigation are considered important to gain insight into polymer processing industry namely, the continuous extrusion of a polymer sheet from a die. Beard-Walters [2] derived the steady two dimensional boundary layer equations for the viscoelastic second grade fluid past a stretching sheet $y = 0$, as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \kappa\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial y}\cdot\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3}\right] \tag{2.2}$$

where u and v are the components of velocity, respectively in the x and y direction; $\nu = \mu/\rho$ is the kinematic viscosity, μ is the viscosity and ρ is the density of the fluid and κ is a positive parameter associated with the viscoelastic fluid.

The boundary conditions for $x \geq 0$, are given by

$$\begin{aligned} &u = Cx \text{ and } v = 0 \text{ at } y = 0 \\ &u \to 0 \text{ as } y \to \infty \end{aligned} \tag{2.3}$$

The sheet is moving in its own plane with a speed proportional to the distance from the origin, C being the constant of proportionality.

In the next section, our aim will be to reduce the momentum equation (2.2) into an ordinary differential equation.

3. Application of scaling group of transformations

To non-dimensionalize the equations, we introduce L as half length of the sheet, $U = CL$ and the Reynolds number $Re = UL/\nu$ and write

$$\overline{x} = \frac{x}{L},\ \overline{y} = \frac{y}{L}\sqrt{Re},\ \overline{u} = \frac{u}{U},\ \overline{v} = \frac{v}{U}\sqrt{Re}. \tag{3.1}$$

Substituting the relations (3.1) in equations (2.1) and (2.2), we obtain

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0 \tag{3.2}$$

$$\overline{u}\frac{\partial \overline{u}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{u}}{\partial \overline{y}} = \frac{\partial^2 \overline{u}}{\partial \overline{y}^2} - \overline{K}\left[\frac{\partial}{\partial \overline{x}}\left(\overline{u}\frac{\partial^2 \overline{u}}{\partial \overline{y}^2}\right) + \frac{\partial \overline{u}}{\partial \overline{y}}.\frac{\partial^2 \overline{v}}{\partial \overline{y}^2} + \overline{v}\frac{\partial^3 \overline{u}}{\partial \overline{y}^3}\right] \tag{3.3}$$

where $\overline{K} = \kappa C/\nu$.

The boundary conditions (2.3) are reduced to

$$\begin{aligned} &\overline{u} = \overline{x} \text{ and } \overline{v} = 0 \text{ at } \overline{y} = 0 \\ &\overline{u} \to 0 \text{ as } \overline{y} \to \infty \end{aligned} \tag{3.4}$$

Let us introduce the stream function ψ as

$$\overline{u} = \frac{\partial \psi}{\partial \overline{y}}, \ \overline{v} = -\frac{\partial \psi}{\partial \overline{x}} \tag{3.5}$$

Clearly, the relations (3.5) satisfy the continuity equation (3.2).

Substituting (3.5) in equation (3.3), we obtain

$$\begin{aligned} &\frac{\partial \psi}{\partial \overline{y}} \cdot \frac{\partial^2 \psi}{\partial \overline{x}\partial \overline{y}} - \frac{\partial \psi}{\partial \overline{x}}\frac{\partial^2 \psi}{\partial \overline{y}^2} \\ &= \frac{\partial^3 \psi}{\partial \overline{y}^3} - \overline{K}\left[\frac{\partial^2 \psi}{\partial \overline{x}\partial \overline{y}} \cdot \frac{\partial^3 \psi}{\partial \overline{y}^3} + \frac{\partial \psi}{\partial \overline{y}}\frac{\partial^4 \psi}{\partial \overline{x}\partial \overline{y}^3} - \frac{\partial^2 \psi}{\partial \overline{y}^2} \cdot \frac{\partial^3 \psi}{\partial \overline{x}\partial \overline{y}^2} - \frac{\partial \psi}{\partial \overline{x}} \cdot \frac{\partial^4 \psi}{\partial \overline{y}^4}\right] \end{aligned} \tag{3.6}$$

The boundary conditions (3.4) are transformed to

$$\begin{aligned} &\frac{\partial \psi}{\partial \overline{y}} = \overline{x}, \ \frac{\partial \psi}{\partial \overline{x}} = 0 \text{ at } \overline{y} = 0 \\ &\frac{\partial \psi}{\partial \overline{y}} \to 0 \text{ as } \overline{y} \to \infty \end{aligned} \tag{3.7}$$

We now introduce the simplified form of Lie-group transformations namely, the scaling group of transformations (see Patel-Timol [8]) as

$$\Gamma : \begin{cases} x^* = e^{\in a}\overline{x}; \ y^* = e^{\in b}\overline{y}; \\ \psi^* = e^{\in c}\psi \\ u^* = e^{\in f}\overline{u}; \ v^* = e^{\in g}\overline{v} \end{cases} \tag{3.8}$$

(3.8) may be considered as a point-transformation which transforms coordinates $(\overline{x}, \overline{y}, \psi, \overline{u}, \overline{v})$ to the coordinates $(x^*, y^*, \psi^*, u^*, v^*)$.

Substituting (3.8) into (3.6), we obtain

$$
\begin{aligned}
& e^{\in(a+2b-2c)}\left[\frac{\partial\psi^*}{\partial y^*}\cdot\frac{\partial^2\psi^*}{\partial x^*\partial y^*}-\frac{\partial\psi^*}{\partial x^*}\cdot\frac{\partial^2\psi^*}{\partial y^{*2}}\right] \\
&= e^{\in(3b-c)}\frac{\partial^3\psi^*}{\partial y^{*3}}-\overline{K}e^{\in(a+4b-2c)}\left[\frac{\partial^2\psi^*}{\partial x^*\partial y^*}\cdot\frac{\partial^3\psi^*}{\partial y^{*3}}\right. \\
&\quad \left.+\frac{\partial\psi^*}{\partial y^*}\cdot\frac{\partial^4\psi^*}{\partial x^*\partial y^{*3}}-\frac{\partial^2\psi^*}{\partial y^{*2}}\cdot\frac{\partial^3\psi^*}{\partial y^{*2}\partial x^*}-\frac{\partial\psi^*}{\partial x^*}\cdot\frac{\partial^4\psi^*}{\partial y^{*4}}\right]
\end{aligned}
\tag{3.9}
$$

In order that the system will remain invariant under the group of transformation Γ, we would have the following relations among the transformation parameters, namely

$$a+2b-2c=3b-c=a+4b-2c \tag{3.10}$$

From (3.10), we obtain easily $b=0$ and $a=c$. In view of this, the boundary conditions are transformed to

$$
\begin{gathered}
\frac{\partial\psi^*}{\partial y^*}=x^*;\ \frac{\partial\psi^*}{\partial x^*}=0 \text{ at } y^*=0 \\
\frac{\partial\psi^*}{\partial y^*}\to 0 \text{ as } y^*\to\infty
\end{gathered}
\tag{3.11}
$$

with the additional conditions $a=c=f$ and $g=0$. Thus the set Γ reduces to a one-parameter group of transformations :

$$x^*=e^{\in a}\overline{x},\ y^*=\overline{y},\ \psi^*=e^{a\in}\psi,\ u^*=e^{a\in}\overline{u},\ v^*=\overline{v} \tag{3.12}$$

Absolute invariants :

First we find an absolute invariant, which is a function of the independent variable, namely $\eta=\overline{y}\,\overline{x}^s$

For this purpose, we write

$$x^*=A\overline{x},\ A=e^{\in a}\,;\ y^*=A^{b/a}\overline{y}\,;\ \psi^*=A^{c/a}\psi \tag{3.13}$$

To establish $y^*x^{*s}=\overline{y}\,\overline{x}^s$, we have

$$y^*x^{*s}=\overline{y}A^{b/a}\cdot A^s\overline{x}^s=A^{s+\frac{b}{a}}\overline{y}\,\overline{x}^s$$

Putting $s+\frac{b}{a}=0$, we obtain $y^*x^{*s}=\overline{y}\,\overline{x}^s$

Now, since $b = 0$ and $s = 0$, we obtain for the present case $\eta = y^* x^{*0} = y^*$,

$$\text{Thus } \eta = y^* \text{ is an absolute invariant.} \tag{3.14}$$

We now find a second absolute invariant G, which involves the dependent variable ψ. Let us assume that $G = \overline{x}^r \psi$. We will find r such that

$$\overline{x}^r \psi = x^{*^r} \psi^*$$

Now,

$$x^{*^r} \psi^* = A^r \overline{x}^r A^{c/a} \psi = A^{r+\frac{c}{a}} \overline{x}^r \psi$$

Putting $r + \frac{c}{a} = 0$, or, $r = -\frac{c}{a} = -1$, since $c = a$.

Thus, the second absolute invariant G is, given by

$$G = x^{*^{-1}} \psi^*$$

Now, putting $G = F(\eta)$, we can write

$$\psi^* = x^* F(\eta) \tag{3.15}$$

4. Solution of the problem

In view of the relations (3.14) and (3.15), we obtain easily the transformed version of equation (3.9) with $b = 0, a = c$, as

$$F'''(\eta) + F(\eta)F''(\eta) - F'^2(\eta) = \overline{K}[2F'(\eta)F'''(\eta) - F''^2(\eta) - F(\eta)F^{iv}(\eta)] \tag{3.16}$$

It can easily verified that equation (3.16) admits an exact solution

$$F(\eta) = \frac{1}{p}\left(1 - e^{-p\eta}\right), \tag{3.17}$$

where

$$p = (1 - \overline{K})^{-1/2};\ 0 < \overline{K} < 1 \tag{3.18}$$

Remarks : The problem of flow of a viscoelastic second grade fluid has been studied by many authors. Unlike these authors, in the present approach it is shown that the application of scaling group of transformations to the problem is unique. It turns out to be a one-parameter group of transformations which helped, finally in deducing the desired equation. As the scaling-group of transformations originate from Lie's group theory, its application to the problems of hydrodynamics is highly desirable.

References

1. W.F. Ames, *Nonlinear Partial Differential Equations in Engineering*, Academic Press, N.Y. (1965).
2. D.W. Beard and K. Walters, *Elastic-Viscous bound any layer flow*, Proc. Camb. Phil. Soc. **60** (1964), 667-674.
3. G. Birkhoff, *Hydrodynamics*, Dover, N.Y. (1955).
4. C.W. Bluman & J. D, Cole, *Similarity Methods for Differential Equations*, Springer-Verlag N.Y. (1974).
5. A.G. Hansen, *Similarity Analysis of Boundary Value Problems in Engineering*, Prentice Hall, Inc. New Jersey (1964).
6. N.H. Ibragimov, *Lie Group Analysis of Differential Equations,* **Vol. 1**, CRC Press Boca Roton, USA (1994).
7. P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag N.Y. (1986).
8. R.K. Patel and M.G. Timol, *Similarity solutions of MHD boundary layer flow of viscous fluid past a moving continuous flat surface*, Bull. Cal. Math. Soc. **92**(2) (2000), 121-126.
9. K.R. Rajagopal, *Flow of a viscoelastic fluid over a stretching sheet*, Rheol. Acta. **23** (1984), 213-215.
10. K.R. Rajagopal, *A non-similar boundary layer on a stretching sheet in a NonNewtonian fluid with uniform free stream*, J. Math. Phys. Sci. **21** (1987), 189-200.
11. B.C. Sakiadis, *Boundary Layer behavior on continuous solid surfaces*, A.I. Ch. E. J. **7** (1961), 26-28.
12. R. Seshadri and T.Y. Na, *Group Invariance in Engineering Boundary Value Problems*, Springer-Verlag N.Y. (1985).
13. B. Siddappa and M. S. Abel, *Non-Newtonian flow past a stretching sheet*, ZAMP **36** (1985), 890-892.
14. H. Stephani, *Differential Equations: Their solution using Symmetries*, University of Cambridge Press, England (1989).

T. Tapanidis
Department of Mathematics and Physics
Aristotle University of Thessaloniki
Thessaloniki 54006, Greece
E-mail address: `tsagkas@eng.auth.gr`

Gr. Tsagas
Department of Mathematics and Physics
Aristotle University of Thessaloniki
Thessaloniki 54006, Greece

H. P. Mazumdar
Physics and Applied Mathematics Unit
Indian Statistical Institute
Kolkata-700 108, India
E-mail address: `hpm@isical.ac.in`