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MEASURE SOLUTIONS FOR EVOLUTION EQUATIONS WITH DISCONTINUOUS VECTOR FIELDS

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Abstract. In this paper we present some results on the question of existence of generalized or measure valued solutions for semilinear evolution equations on Banach spaces with the nonlinear part being merely measurable and bounded on bounded sets. This admits discontinuities and exponential growth of the nonlinear term. This is a far reaching generalization of the previous results of the author and others.

1. Motivation

Let us consider the evolution equation

$$\begin{aligned} \dot{x} &= Ax + f(x), t \geq 0 \\ x(0) &= \xi \end{aligned} \tag{1.1}$$

in a Banach space E where A is the infinitesimal generator of a C_0-semigroup, $S(t)$, $t \geq 0$, on E and $f : E \longrightarrow E$ is a continuous map. It is well known that if E is finite dimensional, mere continuity of f is good enough to prove the existence of local solutions with possibly finite blow up time. If E is an infinite dimensional Banach space this is no longer true unless the semigroup $S(t)$, $t > 0$ is compact. For example, see [1, Theorem 5.3.6, p172]. In recent years [2,3,4,5,6,9], a generalized notion of solution (measure solution) has been introduced extending the standard notions such as classical, strong, mild and weak. This has made it possible to prove the existence of (generalized) solutions without requiring either of the hypothesis: the Lipschitz property of

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f and the compactness of the semigroup. In all these papers, except [5], it was assumed that f is a continuous map and that it is locally Lipschitz and that A is the generator of a C_0-semigroup on E. In [5] the requirement of local Lipschitz condition for f was removed and relaxed to a much broader condition requiring that f is continuous and bounded on bounded sets. This admits f having polynomial and even exponential growth. In this paper we relax this condition further. Here we require f to be only measurable and bounded on bounded sets. The rest of the paper is organized as follows. In section 2, we present some basic concepts related to finitely additive measures and associated function spaces including the definition of measure solutions. In section 3, we present our new results. In section 4, we discuss the question of uniqueness of measure solutions. The paper is concluded with some open questions.

2. Introduction

For the purpose of formulation of measure solutions, we need the characterization of the dual of the Banach space $L_1(I, X)$ where $I \equiv [0, T]$ is a finite interval of the real line and X is a Banach space. Let X^* denote the dual of X, and $< . >$ the duality pairing of X^* and X. It is well known that if both X and X^* satisfy Radon-Nikodym property (RNP) then the dual of $L_1(I, X)$ is given by $L_\infty(I, X^*)$. See Diestel Jr and Uhal [7]. In general it follows from the theory of "Lifting" [10, Theorem 7 and its Corollary , p94] that the dual of $L_1(I, X)$ is given by $L_\infty^w(I, X^*)$ which is the class of w^*-measurable X^*-valued functions $\{g\}$ with weak forms given by $t \longrightarrow < g(t), x >$ being essentially bounded measurable real valued functions. The space is furnished with the norm $\| g \|_{L_\infty^w(I,X^*)} = \alpha$ where α is the smallest number for which the inequality

$$\text{ess-sup}\{|(g(t), x))|, t \in I\} \leq \alpha \| x \|_X$$

is satisfied.

Let Z denote any topological space and $B_0(Z)$ the space of bounded scalar valued functions on Z with the topology of sup norm given by

$$\| f \| \equiv \sup\{|f(z)|, z \in Z\}.$$

This is a Banach space. However the elements of this space may not be measurable. Let Σ denote a field of subsets of the set Z and let $B(Z) \equiv B(Z, \Sigma)$ denote the class of scalar functions which are uniform limits of characteristic functions of sets from Σ. The space $B(Z)$ is furnished with the same topology as in $B_0(Z)$. An element f of this space is said to be Σ measurable if for every Borel set σ in the range space, the set

$$\{z \in Z : f(z) \in \sigma\} \in \Sigma.$$

The class of all bounded Σ measurable functions is dense in $B(Z)$. It is clear that $B(Z)$ is a closed subspace of $B_0(Z)$ and hence it is also a Banach space. Let $\mathcal{M}_{ba}(Z) \equiv \mathcal{M}_{ba}((Z,\Sigma))$ denote the class of all scalar valued finitely additive measures defined on the algebra Σ. Furnished with the total variation norm, $\mathcal{M}_{ba}(Z)$ is a Banach space.

The following Lemma characterizes the topological dual $B^*(Z)$ of the Banach space $B(Z)$.

Lemma 2.1. *The space $B^*(Z) \cong \mathcal{M}_{ba}(Z)$, that is $B^*(Z)$ is isometrically isomorphic to the space of bounded finitely additive measures on $Z \equiv (Z,\Sigma)$ in the sense that, for every $\ell \in B^*(Z)$, there exists a unique $\mu \in \mathcal{M}_{ba}(Z)$ such that*

$$\ell(f) = \int_Z f(z)\mu(dz), f \in B(Z),$$

and conversely, every $\mu \in \mathcal{M}_{ba}(Z)$ determines a unique continuous linear functional on $B(Z)$.

Proof. see Dunford and Schwartz [8, Theorem IV.5.1, p 258]. $\square$

Let $\Pi_{ba}(Z) \subset \mathcal{M}_{ba}(Z)$ denote the class of finitely additive probability measures furnished with the relative topology. The Banach space $B(Z)$ and its dual $\mathcal{M}_{ba}(Z)$ do not satisfy RNP. Therefore it follows from the characterization result discussed in the introduction that the dual of $L_1(I, B(Z))$ is given by $L^w_\infty(I, \mathcal{M}_{ba}(Z))$ which is furnished with the weak star topology.
We consider the Cauchy problem in a Banach space E,

$$\begin{aligned} &\dot{x} = Ax + f(t,x), t \in I \equiv [0,\tau], \\ &x(0) = x_0 \in E, \end{aligned} \tag{2.1}$$

where A is the generator of a C_0-semigroup $S(t), t \geq 0$, in E, and f is map from $I \times E$ to E. Let $\mathcal{B}$ denote the sigma algebra of Borel subsets of the interval I and Σ a field or algebra of subsets of the set E, generated by closed subsets of E. Our general assumption is that f is a $\mathcal{B} \times \Sigma$ measurable map with values in E.

The following general notion of measure solutions was introduced by the author in [3,4,5,6], where regular bounded finitely additive measures $\mathcal{M}_{rba}(Z)$

were used instead of $\mathcal{M}_{ba}(Z) \supset \mathcal{M}_{rba} \cong (BC(Z))^*$. This generalization allows measurable vector fields in the evolution equations. For an earlier definition of measure solutions, which is somewhat restrictive, see Fattorini [9].

Let $D\phi$ denote the Frechet derivative of $\phi \in BC(E)$ whenever it exists and introduce the class of test functions $\mathcal{F}$, given by

$$\mathcal{F} \equiv \{\phi \in BC(E) : D\phi \text{ exists, } D\phi \in B(E, E^*)\}.$$

Define the operator $\mathcal{A}$ with domain given by

$$\mathcal{D}(\mathcal{A}) \equiv \{\phi \in \mathcal{F} : \mathcal{A}\phi \in B(E)\}$$

where

$$(\mathcal{A}\phi)(t,\xi) = < A^* D\phi(\xi), \xi >_{E^*,E} + < D\phi(\xi), f(t,\xi) >_{E^*,E}, \text{ for } \phi \in \mathcal{D}(\mathcal{A}). \tag{2.4}$$

Note that $\mathcal{D}(\mathcal{A}) \neq \emptyset$, for example, for $\psi \in \mathcal{F}$, the function ϕ given by $\phi(x) \equiv \psi(\lambda R(\lambda, A)x)$, belongs to $\mathcal{D}(\mathcal{A})$ for each $\lambda \in \rho(A)$, the resolvent set of A.

Consider the system (2.1) with A and f as defined above. We shall write

$$(\mathcal{A}(t)\phi)(\xi) \equiv (\mathcal{A}\phi)(t,\xi).$$

Definition 2.2. A measure valued function $\mu \in L^w_\infty(I, \mathcal{M}_{ba}(E))$ is said to be a generalized solution of equation (2.1) if, for every $\phi \in \mathcal{D}(\mathcal{A})$ with $D\phi$ having bounded supports, the following equality holds

$$\mu_t(\phi) = \phi(x_0) + \int_0^t \mu_s(\mathcal{A}\phi)ds, t \in I, \tag{2.5}$$

where

$$\mu_t(\psi) \equiv \int_{E^+} \psi(\xi)\mu_t(d\xi), t \in I.$$

For simplicity of notation we shall use $\mathcal{D}(\mathcal{A})$ to denote the common domain of the operators $\mathcal{A}(t), t \in I$.

3. Existence of measure solution

The following result proves the existence of measure solutions for equation (2.1) under the assumption that f is a bounded $\mathcal{B} \times \Sigma$ measurable map on $I \times E$ with values in E.

Theorem 3.1. *Consider the system* (2.1) *and suppose E is a separable Banach space. Let A be the generator of a C_0-semigroup in E and $f : I \times E \longrightarrow E$ be a bounded $\mathcal{B} \times \Sigma$ measurable map satisfying the following approximation property:*

(ai): *there exists a sequence $\{f_n\}$ such that $f_n(t,x) \in \mathcal{D}(A)$ for $x \in E$ and almost all $t \in I$; and, further, for almost all $t \in I$, and $e^* \in E^*$,*

$$< e^*, f_n(t,x) >_{E^*,E} \longrightarrow < e^*, f(t,x) >_{E^*,E} \text{ for each } x \in E.$$

(aii): *for any $r > 0$, there exists a sequence $\{\alpha_{r,n}\} \in L_1^+(I)$, possibly $\| \alpha_{r,n} \| \to \infty$ as $n \to \infty$, such that*

$$\| f_n(t,x) - f_n(t,y) \| \leq \alpha_{r,n}(t) \| x - y \|, x, y \in B_r$$

where $B_r \subset E$ is a ball of radius r around the origin.
Then, for every $x_0 \in E$, the evolution equation (2.1) *has at least one generalized solution $\mu \in L_\infty^w(I, \mathcal{M}_{ba}(E))$ in the sense of definition* (2.2)*. Further, $\mu \in L_\infty^w(I, \Pi_{ba}(E))$ and it is w^* continuous.*

Proof. Let $\rho(A)$ denote the resolvent set of the operator A and $R(\lambda, A)$ the corresponding resolvent operator for $\lambda \in \rho(A)$. Since A is the infinitesimal generator of a C_0-semigroup there exists a nonnegative number ω such that $(\omega, \infty) \subset \rho(A)$. Let $A_n \equiv nAR(n, A)$ denote the Yosida approximation of A defined for all $n \in \rho(A)$. Now consider the sequence of evolution equations

$$\begin{aligned} \dot{x} &= A_n x + f_n(t,x), t \in I, \\ x(0) &= x_{0,n} \equiv nR(n,A)x_0. \end{aligned} \tag{2.1$_n$}$$

By assumption, f is bounded measurable, and the sequence f_n converges to f in the sense described by hypothesis (ai). Thus $\{f_n\}$ must also be a bounded sequence. Let the common bound be denoted by b_f, that is,

$$\sup\{\| f(t,\xi) \|_E, \| f_n(t,\xi) \|_E, (t,\xi) \in I \times E\} \leq b_f.$$

Since f_n is contained in $D(A)$ and, by assumption (aii), they are locally Lipschitz and the data $x_{0,n} \in D(A)$, it follows from semigroup theory [see 1, p156] that for each $n \in \rho(A)$ this equation has a unique strong solution x_n with values $x_n(t) \in D(A)$ and $\dot{x}_n \in L_1(I,E)$ satisfying the first identity of equation $(2.1)_n$, for almost $t \in I$. Since every strong solution is also a mild solution, x_n must also satisfy the integral equation

$$x_n(t) = S_n(t)x_{0,n} + \int_0^t S_n(t-s) f_n(s, x_n(s)) ds, t \in I, \tag{3.1}$$

where $S_n(t)$, $t \geq 0$, is the semigroup generated by A_n. In fact these are uniformly continuous semigroups since their generators are bounded operators. Since

$$\sup\{\| S(t) \|, t \in I\} \leq M$$

and

$$S_n(t) \longrightarrow S(t)$$

in the strong operator topology in $\mathcal{L}(E)$ uniformly on compact intervals, it follows from uniform boundedness principle that there exists a finite positive number $\tilde{M} \geq M$ such that

$$\sup\{\| S_n(t) \|, t \in I\} \leq \tilde{M}.$$

Hence it follows from equation (3.1) that that there exists a finite positive number $\tilde{r}$ such that

$$\sup\{\| x_n(t) \|, t \in I\} \leq \tilde{M}\{\| x_o \| + b_f T\} \equiv \tilde{r} \; \forall n \in N.$$

Thus for any $r \geq \tilde{r}$, we have $x_n(t) \in B_r \subset E$ for all $t \in I$ and all $n \in N$. Since $\{x_n\}$ is a sequence of strong solutions of equation $(2.1)_n$, it is clear that, for every $\phi \in \mathcal{F}$,

$$\phi(x_n(t)) = \phi(x_{0,n}) + \int_0^t < D\phi(x_n(s)), A_n x_n(s) + f_n(s, x_n(s)) >_{E^*,E} ds. \quad (3.2)$$

Let $\delta_e(\cdot)$ denote the Dirac measure on E with its mass concentrated at the point $e \in E$ and define

$$\lambda_t^n(d\xi) \equiv \delta_{x_n(t)}(d\xi) \text{ and } \lambda_0^n(d\xi) = \delta_{x_{0,n}}(d\xi).$$

Using this notation we can rewrite equation (3.2) in the form

$$\lambda_t^n(\phi) = \lambda_0^n(\phi) + \int_0^t \lambda_s^n(\mathcal{A}_n(s)\phi) ds, t \in I, \quad (3.3)$$

where the operator $\mathcal{A}_n$ is given by

$$(\mathcal{A}_n \phi)(t, \xi) = < A_n^* D\phi(\xi), \xi >_{E^*,E} + < D\phi(\xi), f_n(t, \xi) >_{E^*,E}, \quad (3.4)$$

for $\phi \in \mathcal{D}(\mathcal{A})$. Clearly, for each integer $n \in \rho(A)$, $\lambda^n \in L_\infty^w(I, \Pi_{ba}(E)) \subset L_\infty^w(I, \mathcal{M}_{ba}(E))$ and it follows from our preceding analysis that

$$supp(\lambda_t^n) \subset B_r, \forall t \in I, n \in N.$$

Our concern now is to show that the sequence $\{\lambda^n\}$ has a limit and that the limit is a generalized solution of our original problem. Towords this goal, consider the sequence of linear functionals $\{\ell_n\}$ given by

$$\ell_n(\varphi) \equiv \int_{I\times E} \varphi(t,\xi)\lambda_t^n(d\xi)dt. \tag{3.5}$$

Clearly this is well defined for each $\varphi \in L_1(I, B(E))$ and

$$|\ell_n(\varphi)| \leq \parallel \varphi \parallel_{L_1(I,B(E))}, \forall n \in \rho(A).$$

In other words $\{\ell_n\}$ is a sequence of bounded linear functionals contained in a bounded subset of $(L_1(I, B(E)))^*$ the dual of $L_1(I, B(E))$. Thus it follows from the characterization of the dual space of the Banach space $L_1(I, B(E))$, that the sequence $\{\lambda^n\}$ is confined in a bounded subset of $L_\infty^w(I, \mathcal{M}_{ba}(E))$. Hence by Alaoglu's theorem there exists a subsequence (subnet) of the sequence (net) $\{\lambda^n\}$, relabeled as $\{\lambda^n\}$, and a $\lambda^o \in L_\infty^w(I, \mathcal{M}_{ba}(E))$ such that

$$\lambda^n \xrightarrow{w^*} \lambda^o, \text{ in } L_\infty^w(I, \mathcal{M}_{ba}(E)). \tag{3.6}$$

We must show that λ^o is a measure (generalized) solution of the evolution equation (2.1) in the sense of Definition 2.2. Let $\phi \in D(\mathcal{A})$ with both ϕ and $D\phi$ being continuous and bounded having compact supports which may be different for different ϕ. Define

$$\begin{aligned} (B_n\phi)(\xi) &\equiv< (A_n^* - A^*)D\phi(\xi), \xi >_{E^*,E} \\ (C_n(t)\phi)(\xi) &\equiv (C_n\phi)(t,\xi) \\ &\equiv< D\phi(\xi), f_n(t,\xi) - f(t,\xi) >_{E^*,E}, (t,\xi) \in I \times E. \end{aligned} \tag{3.7}$$

Using these expressions, equation (3.3) can be rewritten as

$$\begin{aligned} \lambda_t^n(\phi) = \lambda_0^n(\phi) &+ \int_0^t \lambda_s^n(\mathcal{A}(s)\phi)ds \\ &+ \int_0^t \lambda_s^n(B_n\phi)ds + \int_0^t \lambda_s^n(C_n(s)\phi)ds, t \in I. \end{aligned} \tag{3.8}$$

Consider the first expression of equation (3.7). Since $A_n \longrightarrow A$ on $D(A)$ in the strong operator topology and, for $\phi \in D(\mathcal{A})$, $D\phi(\xi) \in D(A^*)$, and by our choice $D\phi$ is continuous and bounded having compact support, it is clear that $B_n\phi \in B(E)$ and that

$$(B_n\phi) \xrightarrow{s} 0 \text{ in } B(E).$$

Since $B_n\phi$ is independent of t and I is a finite interval, it is evident that

$$(B_n\phi) \xrightarrow{s} 0 \text{ in } L_1(I, B(E)) \tag{3.9}$$

as $n \to \infty$. Consider the second expression of (3.7). By use of similar arguments and assumption (ai) along with Lebesgue dominated convergence theorem, one can easily verify that

$$(C_n\phi) \xrightarrow{s} 0 \text{ in } L_1(I, B(E)) \tag{3.10}$$

also as $n \to \infty$. Let χ_σ denote the characteristic function of any set $\sigma \in \mathcal{B}$. In view of (3.6), (3.9) and (3.10), we have, for each $t \in I$,

$$\begin{aligned} \int_0^t \lambda_s^n(B_n\phi)ds &\equiv \int_I \chi_{[0,t]}(s)\lambda_s^n(B_n\phi)ds \longrightarrow 0 \\ \int_0^t \lambda_s^n(C_n\phi)ds &\equiv \int_I \chi_{[0,t]}(s)\lambda_s^n(C_n\phi)ds \longrightarrow 0, \end{aligned} \tag{3.11}$$

as $n \to \infty$. Note that for $\phi \in D(\mathcal{A})$ with $D\phi$ having bounded support, it follows from $\mathcal{B}\times\Sigma$ measurability and boundedness of f that $\mathcal{A}\phi \in L_1(I, B(E))$. Hence for the second term on the right hand side of equation (3.8), it follows from (3.6) that

$$\begin{aligned} &\int_0^t \lambda_s^n(\mathcal{A}(s)\phi)ds \equiv \int_I \chi_{[0,t]}(s)\lambda_s^n(\mathcal{A}(s)\phi)ds \\ &\longrightarrow \int_I \chi_{[0,t]}(s)\lambda_s^o(\mathcal{A}(s)\phi)ds = \int_0^t \lambda_s^o(\mathcal{A}(s)\phi)ds, \end{aligned} \tag{3.12}$$

as $n \to \infty$. Since $x_{0,n} \xrightarrow{s} x_0$ in E and ϕ is continuous and bounded we have

$$\phi(x_{0,n}) \longrightarrow \phi(x_0) \tag{3.13}$$

as $n \to \infty$. Thus letting $n \to \infty$ in (3.8), it follows from (3.11), (3.12), (3.13) and (3.6) that

$$\lambda_t^o(\phi) = \lambda_0(\phi) + \int_0^t \lambda_s^o(\mathcal{A}(s)\phi)ds, t \in I, \tag{3.14}$$

where $\lambda_0(\phi) = \delta_{x_0}(\phi) = \phi(x_0)$. Since $\lambda^o \in L^w_\infty(I, \mathcal{M}_{ba}(E))$ and the second term on the right hand side of the above identity is bounded for any $\phi \in D(\mathcal{A})$ and the first term holds for any continuous (even bounded) ϕ, it is clear that equation (3.14) holds for all $\phi \in D(\mathcal{A})$, and not just for only those having compact supports. Since $\lambda^o \in L^w_\infty(I, \mathcal{M}_{ba}(E))$ and it is the (unique) weak star limit of a sequence $\{\lambda^n\}$ from $L^w_\infty(I, \Pi_{ba}(E))$ and that this set is weak star closed we have $\lambda^o \in L^w_\infty(I, \Pi_{ba}(E))$. This proves that λ^o is a generalized solution of the evolution equation (2.1) in the sense of Definition 2.2. The last part of the theorem, asserting w^* continuity, follows readily from the integral expression (3.14). This completes the proof. □

Remark. *We have seen that $supp(\lambda^n_t) \subset B_r$ for all $t \in I$ and for all $n \in N$ where the number $r = r(\tilde{M}, b_f, T, \| x_0 \|)$ is as defined earlier which depends on the parameters as displayed. Using this fact it is easy to verify that the limit has similar support properties. Precisely, we have*

$$supp(\lambda^o_t) \subset \bar{B}_r$$

for all $t \in I$.

The result given above depends on the existence of an approximating sequence $\{f_n\}$ approximating the nonlinear measurable map f. The existence of such an approximating sequence follows from the following result. Thus these assumptions are natural and do not in any way limit the results.

Proposition 3.2. *Suppose E is a separable Banach space, A is a linear* (*generally unbounded*) *operator with domain and range in E having nonempty resolvent set $\rho(A)$ with resolvent denoted by $R(\lambda, A)$. Then, for every bounded $\mathcal{B} \times \Sigma$ measurable map $f = f(t, x)$ which is Lebesgue-Bochner integrable in t on I, uniformly with respect to x in bounded subsets of E, there exists a sequence $\{f_n\}$ satisfying the hypotheses* (ai) *and* (aii) *of Theorem* 3.1.

Proof. By virtue of separability, the Banach space E has a Schauder basis $\{e_i\}$. Corresponding to this basis, let $\{E_n\} \subseteq E$ be an increasing family of n-dimensional subspaces of E and $\{Q_n\}$ the corresponding family of projections of E to E_n. Let $\Lambda_n : E \longrightarrow R^n$ denote the linear map taking each element x of E into it's first n Fourier coefficients. That is $\Lambda_n x = col\{(\ell_i(x)), i = 1, 2, 3 \cdots n\}$, where $\{\ell_i\}$ is a sequence of continuous linear functionals on E with $\| \ell_i \|_{E^*}= 1$ associated with the Schauder basis $\{e_i\}$ of E. We use C^∞ molifiers to construct a smooth family $\{f_n\}$ approximating the given f. Let $n \in N$ and $\rho_n \in C^\infty_0(R^n)$ be a family of C^∞ functions on R^n with compact supports satisfying

$$\rho_n(\xi) \geq 0, \rho_n(\xi) = \rho_n(-\xi), supp(\rho_n) \subseteq \{\xi \in R^n : |\xi|_{R^n} \leq (1/n)\}$$
$$\text{and } \int_{R^n} \rho_n(\xi) d\xi = 1, n \in N.$$

Let $J_n \equiv nR(n,A)$ for $n \in \rho(A) \cap N$ and recall that J_n converges in the strong operator topology to the identity operator in E and that $J_n(E) \subset D(A)$. Define

$$f_n(t,x) \equiv \int_{R^n} J_n f(t, Q_n x - \sum_{i=1}^n \xi_i e_i) \rho_n(\xi) d\xi.$$

By a simple change of variables this can be written as

$$f_n(t,x) \equiv \int_{R^n} J_n f(t, \sum_{i=1}^n \eta_i e_i) \rho_n(\Lambda_n x - \eta) d\eta. \tag{3.15}$$

Since f is a bounded $(\mathcal{B} \times \Sigma)$ measurable map on $I \times E$ and ρ_n has compact support, the integral is well defined and hence the sequence $\{f_n\}$ is well defined satisfying $f_n(t,x) \in D(A)$ for all $(t,x) \in I \times E$. Further, for any $e^* \in E^*$, it follows from the following expression,

$$< f_n(t,x), e^* >_{E,E^*} = \int_{R^n} < J_n f(t, \sum_{i=1}^n \eta_i e_i), e^* >_{E,E^*} \rho_n(\Lambda_n x - \eta) d\eta, \tag{3.16}$$

that, for almost all $t \in I$, and every $x \in E$, $f_n(t,x)$ converges weakly to $f(t,x)$. This is weak star convergence of f_n point wise in $x \in E$. Clearly this also implies uniform convergence on compact subsets of E. Thus the hypothesis (ai) is satisfied. For the local Lipschitz property, note that

$$f_n(t,y) - f_n(t,x) \equiv \int_{R^n} J_n f(t, \sum_{i=1}^n \eta_i e_i) \{\rho_n(\Lambda_n y - \eta) - \rho_n(\Lambda_n x - \eta)\} d\eta. \tag{3.17}$$

Using Lagrange formula applied to the modifier, we have

$$\rho_n(\eta) = \rho_n(\xi) + \int_0^1 (D\rho_n(\xi + \theta(\eta - \xi)), \eta - \xi) d\theta, \eta, \xi \in R^n$$

where D denotes the first Frechet derivative. Taking any ball $B_r \subset E, r > 0$, and using this formula in equation (3.17) one can verify that there exists an $\alpha_{r,n} \in L_1^+(I)$ such that

$$\| f_n(t,y) - f_n(t,x) \|_E \leq \alpha_{r,n}(t) \| y - x \|_E, \forall t \in I,$$

for all $x, y \in B_r$. Indeed, by simple computation one can discover that the function $\alpha_{r,n}$ can be chosen as

$$\alpha_{r,n}(t) \equiv \int_{R^n} \gamma\{\parallel J_n f(t, \sum_{i=1}^{n} \eta_i e_i) \parallel_E\} g_{r,n}(\eta) d\eta$$

where the function $g_{r,n}$ is given by

$$g_{r,n}(\eta) \equiv \sup_{0 \le \theta \le 1; x,y \in B_r} |D\rho_n(\Lambda_n x - \eta + \theta(\Lambda_n y - \Lambda_n x))|_{R^n}$$

and γ is the smallest positive number for which $|\Lambda_n x|_{R^n} \le \gamma \parallel x \parallel_E$ for all $x \in E$ and for all $n \in N$. Since $D\rho_n$ is also a C^∞ function having compact support, it is clear that $g_{r,n}$ vanishes outside a bounded subset of R^n. Thus for the given f which is a bounded $\mathcal{B} \times \Sigma$ measurable map taking values from E, the integral defining the function $\alpha_{r,n}$ is finite almost every where. Hence $\{\alpha_{r,n}\}$ is a well defined sequence of finite (actually bounded) measurable functions. Since by our assumption, $t \longrightarrow f(t, x)$ is Bochner integrable, uniformly with respect to x on bounded subsets of E, it follows from Fubini's theorem that $\alpha_{r,n} \in L_1^+(I)$. Thus hypothesis (aii) holds. This completes the proof. $\square$

Remark. *Note that, for the proof of the previous proposition, we have only used the $\mathcal{B} \times \Sigma$ measurability of f and its uniform boundedness on E. In other words it is not necessary that f be a bounded $\mathcal{B} \times \Sigma$ measurable map on $I \times E$.*

Now we are prepared to prove the existence result for unbounded $\mathcal{B} \times \Sigma$ measurable map f defined on $I \times E$ taking values from E. We prove this result under the assumption that f is $\mathcal{B} \times \Sigma$ measurable and that it is bounded only on bounded subsets of E.

Theorem 3.3. *Let A be the infinitesimal generator of a C_0-semigroup in the Banach space E and $f : I \times E \mapsto E$ is $\mathcal{B} \times \Sigma$ measurable, integrable in t on I uniformly with respect to x on bounded subsets of E, and, for almost all $t \in I$, it is bounded on bounded subsets of E. Then, for each $x_0 \in E$, the evolution equation* (2.1) *has at least one measure solution $\lambda \in L_\infty^w(I, \Pi_{ba}(E))$. Further $t \to \lambda_t$ is w^* continuous.*

Proof. The basic technique is similar to that of [5, Theorem 3.2, p1341]. We give a brief outline. Define for each $\gamma > 0$,

$$f_\gamma(t, x) \equiv f(t, R_\gamma(x))$$

where R_γ is the retraction of the ball $B_\gamma \subset E$, that is,

$$R_\gamma(\xi) \equiv \begin{cases} \xi, & \text{if } \xi \in B_\gamma \\ (\gamma / \parallel \xi \parallel)\xi, & \text{otherwise.} \end{cases}$$

Clearly, f_γ is $\mathcal{B}\times\Sigma$ measurable, and, for each $\xi \in E$, $t \longrightarrow f_\gamma(t,\xi)$ is integrable while for almost all $t \in I$, $\xi \longrightarrow f_\gamma(t,\xi)$ is uniformly bounded on all of E. Thus, for each $\gamma < \infty$, it follows from Theorem 3.1 that the evolution equation

$$\begin{aligned} \dot{x} &= Ax + f_\gamma(t,x), t \in I, \\ x(0) &= x_0, \end{aligned} \tag{3.18$_\gamma$}$$

has at least one measure solution $\lambda^\gamma \in L_\infty^w(I, \Pi_{ba}(E))$. In other words, λ^γ is a measure solution of the evolution equation $(3.18)_\gamma$ satisfying

$$\begin{aligned} \lambda_t^\gamma(\phi) &= \lambda_0^\gamma(\phi) + \int_0^t \lambda_s^\gamma(\mathcal{A}_\gamma\phi)ds, \\ &= \phi(x_0) + \int_0^t \lambda_s^\gamma(\mathcal{A}_\gamma(s)\phi)ds, t \in I, \end{aligned} \tag{3.19}$$

for each $\phi \in D(\mathcal{A}_\gamma)$, with $D\phi$ having bounded support, where the operator $\mathcal{A}_\gamma(t)$ is given by

$$(\mathcal{A}_\gamma(t)\phi)(\xi) \equiv < A^*D\phi(\xi), \xi > + < D\phi(\xi), f_\gamma(t,\xi) > . \tag{3.20}$$

Clearly for $\phi \in D(\mathcal{A})$, the identity (3.20) can be rewritten as

$$\mathcal{A}_\gamma(t)\phi = \mathcal{A}(t)\phi + \mathcal{B}_\gamma(t)\phi, \tag{3.21}$$

where

$$\mathcal{B}_\gamma(t)\phi(\xi) \equiv < D\phi(\xi), f_\gamma(t,\xi) - f(t,\xi) >_{E^*,E} . \tag{3.22}$$

Now for each $\gamma > 0$, the functional ℓ_γ given by

$$\begin{aligned} \ell_\gamma(\psi) &\equiv \int_I \lambda_t^\gamma(\psi)dt \\ &\equiv \int_I \int_E \psi(t,\xi)\lambda_t^\gamma(d\xi)dt, \end{aligned} \tag{3.23}$$

is well defined on $L_1(I, B(E))$. Indeed, for all $\gamma > 0$, we have

$$|\ell_\gamma(\psi)| \leq \parallel \psi \parallel_{L_1(I,B(E))} \tag{3.24}$$

for all $\psi \in L_1(I, B(E))$. Thus $\{\ell_\gamma, \gamma > 0\}$ is a bounded subset of $(L_1(I, B(E)))^*$, and hence by the characterization of the dual spaces, the set $\{\lambda^\gamma, \gamma > 0\}$ is contained in a bounded subset of $L^w_\infty(I, \mathcal{M}_{ba}(E))$. Therefore, again by Alaoglus theorem, there exists a subnet or a generalized subsequence $\{\lambda^k \equiv \lambda^{\gamma_k}\}$ and a $\lambda^o \in L^w_\infty(I, \mathcal{M}_{ba}(E))$ such that

$$\lambda^k \xrightarrow{w^*} \lambda^o \text{ as } k \to \infty \tag{3.25}$$

in $L^w_\infty(I, \mathcal{M}_{ba}(E))$. Defining $\mathcal{A}_k \equiv \mathcal{A}_{\gamma_k}, \mathcal{B}_k \equiv \mathcal{B}_{\gamma_k}$, it follows from (3.19) and (3.21) that

$$\begin{aligned} \lambda^k_t(\phi) &= \lambda^k_0(\phi) + \int_0^t \lambda^k_s(\mathcal{A}_k(s)\phi)ds, \\ &= \phi(x_0) + \int_0^t \lambda^k_s(\mathcal{A}(s)\phi)ds + \int_0^t \lambda^k_s(\mathcal{B}_k(s)\phi)ds, t \in I, \end{aligned} \tag{3.26}$$

for all $\phi \in D(\mathcal{A})$. Since, for almost all $t \in I$,

$$f_k \equiv f_{\gamma_k} \longrightarrow f \text{ as } k \to \infty$$

uniformly on bounded subsets of E, and f_k is integrable in t on I uniformly with respect to x in bounded subsets of E, for each $\phi \in D(\mathcal{A})$ with $D\phi$ having bounded support, it follows from dominated convergence theorem that

$$\mathcal{B}_k\phi \xrightarrow{s} 0 \text{ in } L_1(I, B(E)).$$

This, combined with (3.25), implies that for each $t \in I$,

$$\int_0^t \lambda^k_s(\mathcal{B}_k(s)\phi)ds \longrightarrow 0. \tag{3.27}$$

Similarly, for each $\phi \in D(\mathcal{A})$ having Frechet derivatives with bounded support, $\mathcal{A}\phi \in L_1(I, B(E))$. Thus letting $k \to \infty$ in (3.26), it follows from (3.25), (3.27) that, for each $\phi \in D(\mathcal{A})$ having Frechet derivative with bounded support, we obtain

$$\lambda^o_t(\phi) = \lambda_0(\phi) + \int_0^t \lambda^o_s(\mathcal{A}(s)\phi)ds, t \in I. \tag{3.28}$$

Hence $\lambda^o \in L^w_\infty(I, \mathcal{M}_{ba}(E))$ is a measure solution of the evolution equation (2.1) in the sense of Definition 2.2. Since for each integer $k \in N$, $\lambda^k \in$

$L^w_\infty(I,\Pi_{ba}(E))$ and this set is a weak star closed subset of $L^w_\infty(I,\mathcal{M}_{ba}(E))$, we conclude that $\lambda^o \in L^w_\infty(I,\Pi_{ba}(E))$. The last part of the statement asserting w^*-continuity follows trivially from the expression (3.28). This completes the proof. □

In fact the evolution equation (2.1) has measure solution not only for initial data given by a Dirac measure but also for any initial data described by a measure $\pi_0 \in \Pi_{ba}(E)$. This is proved in the following corollary.

Corollary 3.4. *The conclusions of Theorems* 3.1 *and Theorem* 3.3 *remain valid for any initial state* $\pi_0 \in \Pi_{ba}(E)$.

Proof. For any $\xi \in E$, let $\lambda^\xi \in L^w_\infty(I,\Pi_{ba}(E))$ denote the measure solution (see Definition 2.2) of evolution equation (2.1) with initial state given by $x_0 = \xi$, that is $\lambda_0 = \delta_\xi$. Then clearly it follows from either of the above results (Theorem 3.1 or Theorem 3.3) that λ^ξ satisfies the following functional equation,

$$\lambda^\xi_t(\phi) = \phi(\xi) + \int_0^t \lambda^\xi_s(\mathcal{A}(s)\phi)ds, t \in I, \tag{3.29}$$

for every $\phi \in D(\mathcal{A})$ with $D\phi$ having bounded supports. Since $t \longrightarrow \lambda^\xi_t$ is weak star continuous, it is clear that for any $\phi \in D(\mathcal{A})$ with bounded support, $t \longrightarrow \lambda^\xi_t(\phi)$ is continuous and bounded. Also for fixed $t \in I$, and $\phi \in D(\mathcal{A})$, $\xi \longrightarrow \lambda^\xi_t(\phi)$ is a bounded Σ measurable function on E. This follows from the fact that the function $\xi \longrightarrow \lambda^\xi_t(\phi)$ is the point wise limit of a sequence of continuous and hence Σ measurable functions $\xi \longrightarrow \lambda^{n,\xi}_t(\phi)$, here $\lambda^{n,\xi}$ is the measure solution of equation $(2.1)_n$ (see Theorem 3.1) corresponding to the initial data $x_0 = \xi$. Integrating both sides of the identity (3.29) with respect to the measure π_0 and using Fubini's theorem we obtain

$$\lambda_t(\phi) = \pi_0(\phi) + \int_0^t \lambda_s(\mathcal{A}(s)\phi)ds, t \in I, \tag{3.30}$$

where

$$\begin{aligned}\lambda_t(\phi) &\equiv \int_E \left(\int_E \phi(\eta)\lambda^\xi_t(d\eta)\right)\pi_0(d\xi), t \in I,\\ &= \int_E \lambda^\xi_t(\phi)\pi_0(d\xi), t \in I.\end{aligned} \tag{3.31}$$

Since this last integral is finite for any bounded Σ measurable function ϕ, taking $\phi = \chi_\Gamma$, the characteristic function of any Σ measurable set $\Gamma \subset E$, we have

$$\lambda_t(\Gamma) = \int_E \lambda_t^\xi(\Gamma)\pi_0(d\xi). \tag{3.32}$$

For $\Gamma = E, \lambda_t(E) = 1, t \in I$. Hence λ given by (3.31) satisfies the functional equation (3.30) and is an element of the set $L^w_\infty(I, \Pi_{ba}(E))$, and therefore a measure solution of the evolution equation (2.1) corresponding to the initial data given by the distribution π_0. This completes the proof. □

4. Differential equations on the space of measures

Note that in view of our notion of measure solution (see Definition 2.2) and the preceding results, we can reformulate our original Cauchy problem defined on the Banach space E, as a Cauchy problem on the Banach space of finitely additive measures $\mathcal{M}_{ba}(E)$ as follows:

$$\begin{aligned} (d/dt)\mu_t &= \mathcal{A}^*(t)\mu_t, t \geq 0, \\ \mu_0 &= \pi_0. \end{aligned} \tag{4.1}$$

This of course covers the original Cauchy problem as a special case. According to our existence results, we have seen that this equation has solution in the weak sense as implied by our Definition 2.2. Hence it follows from these results, (Theorem 3.1, Theorem 3.3), that for each initial data $\pi_0 \in \Pi_{ba}(E) \subset \mathcal{M}_{ba}(E)$, evolution equation (4.1) has at least one solution $\mu \in L^w_\infty(I, \Pi_{ba}(E)) \subset L^w_\infty(I, \mathcal{M}_{ba}(E))$ which is weak star continuous. Consequently there exists a weak star continuous transition operator $U^*(t,s), 0 \leq s \leq t < \infty$, which is a family of bounded linear operators on the Banach space $\mathcal{M}_{ba}(E)$ defining the evolution of the measure solution

$$\mu_t = U^*(t,0)\pi_0. \tag{4.2}$$

Since for any pair $(s,t) \in I$ satisfying $0 \leq s \leq t$, $U^*(t,s)$ is a bounded linear operator on the Banach space $\mathcal{M}_{ba}(E)$, it is necessarily continuous. In particular, for $s = 0$, let $\pi_0 \in \Pi_{ba}(E)$ and let μ denote the corresponding solution of equation (4.1). Then for any $t \in I$, there exists a positive constant C_t, independent of π_0, such that

$$\| \mu_t \|_{\mathcal{M}_{ba}(E)} = \| U^*(t,0)\pi_0 \|_{\mathcal{M}_{ba}(E)} \leq C_t \| \pi_0 \|_{\mathcal{M}_{ba}(E)} .$$

So far we have not discussed the question of uniqueness of solutions. This is of course equivalent to the question of uniqueness of the transition operator

$U^*(t,s)$. In the absence of uniqueness, the transition operator may not satisfy the expected evolution property

$$U^*(t,r)U^*(r,s) = U^*(t,s), 0 \le s \le r \le t < \infty. \tag{4.3}$$

In regards to this question of uniqueness we have only partial result. For simplicity let us consider the time invariant case. Suppose $\mathcal{A}$ is a discrete spectral operator with domain and range in the Banach space $B(E)$. This implies that it has a countably infinite set of eigen values each with finite multiplicity which may be ordered as follows:

$$\{\cdots \gamma_{n+2} \le \gamma_{n+1} \le 0 \le \gamma_n \le \gamma_{n-1} \le \cdots \gamma_1\}.$$

In this ordering each eigen value is repeated as many times as required by its multiplicity. Let $\{\varphi_i, i \in N\}$ denote the corresponding eigen functions and suppose this set is complete in $B(E)$. Consider the bounded case (Theorem 3.1). Here we saw that the measure solution has bounded support $\bar{B}_r(E)$, for some finite positive number r which depends on the initial state. Choose any two initial states $x_0 = \{\xi, \eta\}$ and let $\bar{B}_\alpha(E)$ denote the closed ball of radius α containing the supports of both the solutions $\{\lambda_t^\xi, \lambda_t^\eta\}$. Define the measure $\nu_t^{\xi,\eta} \equiv \lambda_t^\xi - \lambda_t^\eta$. Note that $\nu^{\xi,\eta} \in L_\infty(I, \mathcal{M}_{ba}(E))$ and not in the set $L_\infty(I, \Pi_{ba}(E))$. Using the eigen function φ_i in equation (3.14) corresponding to two distinct initial states $\{\xi, \eta\}$ and taking the difference, we obtain

$$\begin{aligned} \nu_t^{\xi,\eta}(\varphi_i) &= \varphi_i(\xi) - \varphi_i(\eta) + \int_0^t \nu_s^{\xi,\eta}(\mathcal{A}\varphi_i) ds \\ &= \varphi_i(\xi) - \varphi_i(\eta) + \int_0^t \gamma_i \nu_s^{\xi,\eta}(\varphi_i) ds, t \in I. \end{aligned} \tag{4.4}$$

By virtue of Gronwall's inequality, it follows from this that

$$|\nu_t^{\xi,\eta}(\varphi_i)| \le |\varphi_i(\xi) - \varphi_i(\eta)| exp\{|\gamma_i| T\}, t \in I. \tag{4.5}$$

Hence as $\xi \longrightarrow \eta$, $\nu_t^{\xi,\eta}(\varphi_i) \longrightarrow 0$ for all $t \in I$. This is true for every eigen function φ_i. Since this set is complete in $B(E)$, we conclude that $\nu_t^{\xi,\eta}(\varphi) \longrightarrow 0$ as $\xi \longrightarrow \eta$ for all $t \in I$ and every $\varphi \in B(E)$. This proves uniqueness of solutions. Similarly, let $\pi_1, \pi_2 \in \Pi_{ba}(E)$ be two initial conditions for the system (4.1) with $\mathcal{A}$ assumed time invariant. Suppose both have bounded supports. Let $\mu^1, \mu^2 \in L_\infty^w(I, \Pi_{ba}(E))$ be the corresponding solutions. Clearly for bounded f the solutions also have bounded supports see Theorem 3.1 and the remark following it. Thus, again using the eigen functions, one can verify that

$$|\mu_t^1(\varphi_i) - \mu_t^2(\varphi_i)| \leq |\pi_1(\varphi_i) - \pi_2(\varphi_i)| exp\{|\gamma_i|T\}, t \in I. \tag{4.6}$$

Again by virtue of completeness of the eigen functions, it follows from this inequality that if $\pi_1 \longrightarrow \pi_2$ in the weak star sense in $\Pi_{ba}(E)$, then $\mu^1 \longrightarrow \mu^2$ in the weak star topology of $L_\infty^w(I, \mathcal{M}_{ba}(E))$. This implies uniqueness of solutions as well as continuous dependence on the initial data.

In the unbounded case, the same argument can be used to prove uniqueness of solution for each of the truncated problems

$$\begin{aligned} (d/dt)\mu_t &= \mathcal{A}_\gamma^*(t)\mu_t, t \geq 0, \\ \mu_0 &= \pi_0, \end{aligned} \tag{4.7}$$

associated with the truncated operator $\mathcal{A}_\gamma, 0 < \gamma < \infty$, (see Theorem 3.3) provided π_0 has bounded support. Since uniqueness holds for each finite γ and the solution for the unbounded case is given by the w^*-limit of the net $\{\mu^\gamma\}$ (if required a subnet) , uniqueness for the unbounded case follows. In view of this uniqueness result, we are now assured that the transition operator $U^*(t,s)$ is unique and that it satisfies the evolution property (4.3). Thus for the evolution equation

$$\begin{aligned} (d/dt)\mu_t &= \mathcal{A}^*\mu_t, t \geq s > 0, \\ \mu_s &= \nu, \end{aligned} \tag{4.8}$$

starting at time s, the solution is given by $\mu_t = U^*(t,s)\nu, t \geq s$.

Remark. *We have seen that our solutions are only finitely additive bounded measures. By use of the method of compactification [6], one can extend the solution measures to $E^+ \equiv \beta E$, the Stone-Cech compactification of E. With this extension the solutions are countably additive measure valued functions on the sigma algebra $\sigma(\Sigma)$ induced by the algebra Σ.*

Some Comments and Open Questions: (1): The uniqueness of measure solutions is based on our assumption that the unbounded operator $\mathcal{A}$ is spectral. At this time we do not have a proof of this. (2): It would be interesting to find an alternate method of proof for uniqueness without imposing spectral assumption. (3): We believe that our results can be easily extended to some quasilinear problems as found in [5]. (4): In fact these results can be extended in several fronts like impulsive and stochastic systems of the form

$$dx = Axdt + f(t,x)dt + C(t,x)\nu(dt) + G(t,x)dW, t \geq 0,$$

where ν is a vector measure, W is a cylindrical Brownian motion, and $\{C, G\}$ are suitable $\mathcal{B} \times \Sigma$ measurable operator valued functions. (5): The results given in this paper can be used in control problems involving discontinuous vector fields.

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