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PROPERTIES OF CERTAIN NEW CLASSES OF ANALYTIC FUNCTIONS

Khalida Inayat Noor

Abstract. Let $f_n(z) = \frac{z}{(1-z)^{n+1}}$, $n \in N_0$ and let $f_n^{(-1)}$ be defined such that $f_n \star f_n^{(-1)} = \frac{z}{1-z}$, where $\star$ denotes convolution (Hadamard product). Using the operator $I_n f = f_n \star f_n^{(-1)}$, introduced by Noor, we define some classes of analytic functions in unit disk E and study their properties. Some inclusions relationships, sharp coefficient bounds and radius problems are investigated.

1. Introduction

Let $\mathcal{A}$ denote the class of functions

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{1.1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. Let P_k be the class of functions p defined in E and with representation

$$p(z) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t), \tag{1.2}$$

where $\mu(t)$ is a function with bounded variation on $[-\pi, \pi]$ and it satisfies the conditions

$$\int_{-\pi}^{\pi} d\mu(t) = 2, \qquad \int_{-\pi}^{\pi} |d\mu(t)| \leq k. \tag{1.3}$$

We note that $k \geq 2$ and $P_2 = P$ is the class of analytic functions with positive real part in E with $p(0) = 1$. From the integral representation (1.2), it is

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immediately clear that $p \in P_k$ if, and only if, there are analytic functions $p_1, p_2 \in P$ such that

$$p(z) = (\frac{k}{4} + \frac{1}{2})p_1(z) - (\frac{k}{4} - \frac{1}{2})p_2(z). \tag{1.4}$$

We define the Hadamard product or convolution of two analytic functions

$$f(z) = \sum_{m=0}^{\infty} a_m z^{m+1} \quad \text{and} \quad g(z) = \sum_{m=0}^{\infty} b_m z^{m+1}$$

as

$$(f \star g)(z) = \sum_{m=0}^{\infty} a_m b_m z^{m+1}.$$

Denote $D^n : \mathcal{A} \longrightarrow \mathcal{A}$ be the operator defined by

$$\begin{aligned} D^n f &= \frac{z}{(1-z)^{n+1}} \star f, \quad n = 0, 1, 2, \dots \\ &= z + \sum_{m=2}^{\infty} \frac{(m+n-1)!}{(n!)(m!)} a_n z^m. \end{aligned}$$

We note that $D^0 f(z) = f(z)$, $D^1 f(z) = zf'(z)$ and $D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}$.

The symbol $D^n f$ is called the nth order Ruscheweyh derivative of f. Analogous to $D^n f$, Noor [4] and Noor and Noor [5] defined an integral operator $I_n : \mathcal{A} \longrightarrow \mathcal{A}$ as follows

Let $f_n(z) = \frac{z}{(1-z)^{n+1}}$ and let $f_n^{(-1)}$ be defined such that

$$f_n(z) \star f_n^{(-1)}(z) = \frac{z}{1-z}. \tag{1.5}$$

We note that

$$I_n f = f_n^{(-1)} \star f = \left[\frac{z}{(1-z)^{n+1}} \right]^{(-1)} \star f. \tag{1.6}$$

Note that $I_0 f(z) = zf'(z)$ and $I_1 f(z) = f(z)$.

Also, for $I_n f$, we have the identity [4],

$$(n+1)I_n f - nI_{n+1} f = z(I_{n+1} f)'.$$

The integral operator I_n has also been studied in [1], [2], and [3].

A function $f \in \mathcal{A}$ belongs to B_σ of prestarlike functions of order σ if and only if, for $z \in E$,

$$Re\frac{f(z)}{zf'(0)} > \frac{1}{2}, \quad \text{for} \quad \sigma = 1,$$

and

$$\frac{z}{(1-z)^{2(1-\sigma)}} \star f(z) \in S^\star(\sigma), \quad 0 \le \sigma < 1,$$

where $S^\star(\sigma)$ is the classes of starlike functions g with $Re\frac{zg'(z)}{g(z)} > \sigma$ and $S^\star(0) = S^\star$.

We now have the following.

Definition 1.1. Let $f \in \mathcal{A}$. Then, for $\alpha \ge 0, z \in E$, $f \in T_\alpha(k)$ if, and only if,

$$\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\} \in P_k.$$

Definition 1.2. Let $f \in \mathcal{A}$. Then $f \in T_\alpha(k, n)$ if, and only if, $I_n f \in T_\alpha(k)$ for $\alpha \ge 0$, $z \in E$.

2. Preliminary Results

We give here two basic results which we shall need later on. For the proofs of both, we refer to [6].

Lemma 2.1. *If p is analytic in E and $p(0) = 1$ and $Rep(z) > \frac{1}{2}$, $z \in E$, then for any function F, analytic in E, the function $p \star F$ takes values in the convex hull of the image of E under F.*

Lemma 2.2. *Letf be a prestarlike function of order $\sigma(\sigma \le 1)$, and let g be a starlike function of order σ. Then the generalized convolution operator*

$$\wedge F = \frac{f \star gF}{f \star g}$$

is a convexity preserving operator.

3. Main Results

Theorem 3.1. *The class $T_\alpha(k, n)$ is a convex set.*

Proof. Let $f, g \in T_\alpha(k, n)$ and let, for $0 \le \lambda < 1$,

$$F(z) = \lambda f(z) + (1-\lambda)g(z).$$

Then

$$\begin{aligned}(1-\alpha)\frac{I_nF}{z}+\alpha(I_nF)' &= \lambda[\alpha\frac{I_nf}{z}+(1-\alpha)(I_nf)']+(1-\lambda)[\alpha\frac{I_ng}{z}+(1-\alpha)(I_ng)'] \\ &= \lambda h_1+(1-\lambda)h_2 = h,\end{aligned}$$

$h_1, h_2 \in P_k$ and since P_k is a convex set, $h \in P_k$ and hence this proves the result. □

Theorem 3.2. *Let $f \in T_\alpha(k,n)$. Then F defined by*

$$F(z) = \frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt, \quad Re c > 0$$

also belongs to $T_\alpha(k,n)$.

Proof. Let $G = \phi \star f$, $\phi \in C$, where C is the class of convex univalent functions. Now

$$I_nG = \phi \star I_nf,$$

and

$$\begin{aligned}(1-\alpha)\frac{I_nG}{z}+\alpha(I_nG)' &= (1-\alpha)\frac{(\phi\star I_nf)}{z}+\alpha(\phi\star I_nf)' \\ &= \frac{\phi}{z}\star[(1-\alpha)\frac{I_nf}{z}+\alpha(I_nf)'] \\ &= \frac{\phi}{z}\star p, \quad p \in P_k \\ &= \frac{\phi}{z}\star[(\frac{k}{4}+\frac{1}{2})p_1-(\frac{k}{4}-\frac{1}{2})p_2], \quad p_1,p_2 \in P \\ &= (\frac{k}{4}+\frac{1}{2})(\frac{\phi}{z}\star p_1)-(\frac{k}{4}-\frac{1}{2})(\frac{\phi}{z}\star p_2)\end{aligned}$$

Since ϕ is convex, $Re\frac{\phi(z)}{z} > \frac{1}{2}$ for $z \in E$. Thus, using Lemma 2.1, we note that $G \in T_\alpha(k,n)$.

Now we can write

$$F(z) = \phi_c \star f,$$

where ϕ_c is given by

$$\phi_c(z) = \sum_{m=1}^{\infty}\frac{1+c}{m+c}z^m, \quad Re c > 0$$

and ϕ_c is convex in E. Hence $F \in T_\alpha(k,n)$. □

Theorem 3.3. *Let $f \in T_\alpha(k,n)$, $0 < \alpha < 1$. Then $f \in T_1(k,n)$ for $|z| < R$, where R is the radius of the largest disk centered at the origin for which $Req'(z) > \frac{1}{2}$, $q(z)$ is defined by*

$$q(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z \frac{t^{\frac{1}{\alpha}-1}}{1-t} dt, \quad (\alpha > 0) \tag{3.1}$$

andR is given by the smallest root of the equation

$$\frac{(\frac{2}{\alpha} - 1 - r)}{1+r} - \frac{2}{\alpha}(\frac{1}{\alpha} - 1) \int_0^1 \frac{t^{\frac{1}{\alpha}-1}}{1-tr} dt = 0. \tag{3.2}$$

This result is sharp.

Proof. Since $f \in T_\alpha(k,n)$, we can write

$$I_n f = q \star zp, \qquad p \in P_k.$$

This implies

$$(I_n f)' = \frac{zp \star zq'}{z} = \frac{zp \star zq'}{z \star zq'}.$$

Let $zq' = h$ and so $h' = q' + zq''$. It is easy to check that $q'(0) = 1$. Therefore, for $Req'(z) > \frac{1}{2}$, we see that

$$Re \frac{h(z)}{zh'(0)} > \frac{1}{2}, \quad \text{for} \quad |z| < R.$$

Thus h is a prestralike function of order $\sigma = 1$.

Now

$$\begin{aligned}(I_n f)'(z) &= \frac{zp(z) \star zq'(z)}{z \star zq'(z)} \\ &= (\frac{k}{4} + \frac{1}{2}) \frac{zp_1(z) \star zq'(z)}{z \star zq'(z)} - (\frac{k}{4} - \frac{1}{2}) \frac{zp_2(z) \star zq'(z)}{z \star zq'(z)}, \quad p_1, p_2 \in P.\end{aligned}$$

Using Lemma 2.2 on $\frac{zp_i(z) \star zq'(z)}{z \star zq'(z)}$, $i = 1, 2$, we see that $(I_n f)' \in P_k$ for $|z| < R$ and so$f \in T_1(k,n)$ for$|z| < R$. To find radius R, we proceed as follows.

For $\alpha > 0$, $q(z)$ is given by (3.1), where the powers are meant as principal values. We have

$$q'(z) = \frac{1}{\alpha(1-z)} - \frac{1}{\alpha}(\frac{1}{\alpha} - 1)z^{1-\frac{1}{\alpha}} \int_0^z \frac{t^{\frac{1}{\alpha}-1}}{1-t} dt.$$

The function q' is analytic in E, $q'(0) = 1$ and

$$2q'(z) - 1 = \frac{2 - \alpha + \alpha z}{\alpha(1-z)} - \frac{2}{\alpha}(\frac{1}{\alpha} - 1) \int_0^1 \frac{t^{\frac{1}{\alpha}-1}}{1-tz} dt.$$

Therefore $Re q'(z) > \frac{1}{2}$ for $|z| < R$, where R is the smallest positive root of (3.2). The sharpness of the result follows from the function $f_0 \in T_\alpha(k, n)$ defined by

$$I_n f_0 = q(z) \star zp(z) \tag{3.3}$$

with

$$p(z) = (\frac{k}{4} + \frac{1}{2})\frac{1+z}{1-z} - (\frac{k}{4} - \frac{1}{2})\frac{1-z}{1+z}.$$

□

Theorem 3.4. *Let $f \in T_\alpha(k, n)$, $\alpha > 0$ and be given by* (1.1). *Then*

$$|a_m| \leq \frac{k}{1 - \alpha + \alpha m} \frac{(m + n -)!}{n!m!}.$$

This result is sharp.

Proof. Since $f \in T_\alpha(k, n)$, we have

$$(1-\alpha)\frac{I_n f(z)}{z} + \alpha(I_n f)'(z) = p(z), \quad p \in P_k. \tag{3.4}$$

Let p be given by $p(z) = 1 + \sum_{m=1}^{\infty} c_m z^m$, then it is known that

$$|c_m| \leq k. \tag{3.5}$$

Equating the coefficient of z^{m-1} in (3.4) and using (3.5), we have

$$(1 - \alpha + \alpha m)\frac{(n!)(m!)}{(m + n - 1)!}|a_m| \leq k,$$

and the required result follows.

The function f_0 defined by (3.3) shows that these coefficient bounds are best possible. □

Theorem 3.5. *Let $f \in T_0(k,n)$. Then $f \in T_\alpha(k,n)$ for $|z| < r_\alpha$, where*

$$r_\alpha = \frac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}}, \quad 0 < \alpha < 1, \quad \alpha \neq \frac{1}{2}. \tag{3.6}$$

Proof. Let

$$\phi_\alpha(z) = (1-\alpha)\frac{f(z)}{z} + \alpha f'(z).$$

Then

$$I_n\phi_\alpha(z) = (\frac{\Psi_\alpha(z)}{z}) \star (\frac{I_n f(z)}{z}),$$

where

$$\Psi_\alpha(z) = (1-\alpha)\frac{z}{1-z} + \alpha\frac{z}{(1-z)^2}0 = z + \sum_{m=1}^{\infty}(1+(m-1)\alpha)z^m.$$

Now Ψ_α is convex for $|z| < r_\alpha$, r_α is given by (3.6) and this value is exact. Consequently, for $|z| < r_\alpha$, $Re\frac{\Psi_\alpha(z)}{z} > \frac{1}{2}$. Hence, since $f \in T_0(k,n)$, $\frac{I_n f}{z} \in P_k$ and applying the technique used in the proof of Theorem 3.2, we conclude that $f \in T_\alpha(k,n)$ for $|z| < r_\alpha$. □

Theorem 3.6. *For $0 \leq \alpha_2 < \alpha_1$, $T_{\alpha_1}(k,n) \subset T_{\alpha_2}(k,n)$.*

Proof. For $\alpha_2 = 0$, the proof is immediate. Let $\alpha_2 > 0$ and let $f \in T_{\alpha_1}(k,n)$. Then

$$\begin{aligned}(1-\alpha_2)\frac{I_n f}{z} + \alpha_2(I_n f)' &= \frac{\alpha_2}{\alpha_1}\left[(\frac{\alpha_1}{\alpha_2} - 1)\frac{I_n f}{z} + (1-\alpha_1)\frac{I_n f}{z} + \alpha_1(I_n f)'\right] \\ &= (1 - \frac{\alpha_2}{\alpha_1})\frac{I_n f}{z} + \frac{\alpha_2}{\alpha_1}\left[(1-\alpha_1)\frac{I_n f}{z} + \alpha_1(I_n f)'\right] \\ &= (1 - \frac{\alpha_2}{\alpha_1})H_1 + \frac{\alpha_2}{\alpha_1}H_2, \quad (\frac{\alpha_2}{\alpha_1} < 1)\end{aligned}$$

$H_1, H_2 \in P_k$ and this implies $[(1-\frac{\alpha_2}{\alpha_1})H_1 + \frac{\alpha_2}{\alpha_1}H_2]$ also belongs to P_k. Thus $f \in T_{\alpha_2}(k,n)$. □

Theorem 3.7. *$T_\alpha(k,n) \subset T_\alpha(k,n+1)$.*

Proof. Let$f \in T_\alpha(k,n)$. Then

$$I_n f = q \star zp, \quad p \in P_k$$

and q is convex and is given by (3.1). Using Lemma 2.1, we note that $(\frac{q}{z}\star p) \in P_k$. From identity (1.7), we can write

$$I_n f = \frac{n}{n+1} I_{n+1} f + \frac{1}{n+1} z(I_{n+1} f)'. \tag{3.7}$$

From (3.1) and (3.7) with $\alpha = \frac{1}{n+1}$, we can write

$$I_n f = q_n \star I_{n+1} f.$$

Since $f \in T_\alpha(k,n)$, it implies that $f \in T_0(k,n)$ and therefore $\frac{I_n f}{z} \in P_k$. Thus $\frac{1}{z}(q_n \star I_{n+1} f) \in P_k$. Set

$$I_{n+1} f = q \star zH = (\frac{k}{4}+\frac{1}{2})(q \star zh_1) - (\frac{k}{4}-\frac{1}{2})(q \star zh_2).$$

We want to show that $h_i \in P$, $i = 1, 2$. Now

$$q_n \star I_{n+1} f = (\frac{k}{4}+\frac{1}{2})(q \star q_n \star zh_1) - (\frac{k}{4}-\frac{1}{2})(q \star q_n \star zh_2).$$

Since $\frac{1}{z}(q_n \star I_{n+1} f) \in P_k$, it implies that $\frac{1}{z}(q \star q_n \star zh_i) \in P$, $i = 1, 2$. q, q_n are both are convex, so $q \star q_n = \phi$ is also convex, see [6]. Therefore $\frac{1}{z}(\phi \star zh_i) \in P$ and this implies $h_i \in P$, $i = 1, 2$. This proves that $H \in P_k$ and hence $f \in T_\alpha(k, n+1)$. Hence the proof. □

Let $f \in \mathcal{A}$ and let

$$F(z) = L(f) = \frac{n+1}{z^n} \int_0^z t^{n-1} f(t)dt. \tag{3.8}$$

Then

$$\begin{aligned} F(z) = Lf(z) &= (z + \sum_{m=0}^{\infty} \frac{n+1}{n+m+1}) \star f(z) \\ &= [z {}_2F_1(1, n+1; n+2, z)] \star f(z), \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function.

This implies

$$I_n F(z) = I_n L(f(z)) = I_{n+1} f(z)$$

and we have the following result.

Theorem 3.8. *Let* $f \in T_\alpha(k, n+1)$. *Then* F, *defined by* (3.8), *belongs to* $T_\alpha(k,n)$, *for* $z \in E$.

We now prove a radius problem

Theorem 3.9. *Let* F, *defined by* (3.8), *belong to* $T_\alpha(k,n)$. *Then* $f \in T_\alpha(k,n)$ *for* $|z| < r_n$, *where the exact value of* r_n *is given by*

$$r_n = \frac{(1+n)}{2+\sqrt{3+n^2}}. \tag{3.9}$$

Proof. We can write

$$F(z) = \Psi_n(z) \star f(z), \quad \Psi_n(z) = \sum_{j=1}^{\infty} \frac{n+j}{n+1} z^j,$$

and Ψ_n is convex for $|z| < r_n = \frac{1+n}{2+\sqrt{3+n^2}}$. Thus

$$I_n f = f_n^{(-1)} \star \Psi_n \star f = \Psi_n \star I_n f.$$

Since $F \in T_\alpha(k,n)$, it implies that $\Psi_n \star I_n f \in T_\alpha(k,n)$ and, since Ψ_n is convex in $|z| < r_n$, so $f \in T_\alpha(k,n)$ for $|z| < r_n$, where r_n is given by (3.9). $\square$

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Khalida Inayat Noor
Mathematics Department,
Lahore University of Management Sciences,
Lahore, Pakistan
E-mail address: khalidanoor@hotmail.com