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ON THE STABILITY OF A MIXED $(n, n-1)$- DIMENSIONAL QUADRATIC FUNCTIONAL EQUATION

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Dedicated to the memory of Professor Donald H. Hyers

Abstract. Let $n \geq 3$ be an integer. In this paper, we investigate the generalized Hyers-Ulam-Rassias stability of a mixed $(n, n-1)$-dimensional quadratic functional equation,

$$(n-2)f(\sum_{j=1}^{n} x_j) \quad + \quad \sum_{i=1}^{n} f(x_i) \quad = \sum_{1 \leq i_1 < \cdots < i_{n-1} \leq n} f(x_{i_1} + \cdots + x_{i_{n-1}})\,.$$

The mixed stability problem was posed in the paper; see [11].

1. Introduction

In 1940, the problem of stability of functional equations was originated by Ulam [16] as follows: Under what condition does there exist an additive mapping near an approximately additive mapping ?

The first partial solution to Ulam's question was provided by D. H. Hyers [5]. Let X and Y are Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Hyers showed that if a function $f : X \to Y$ satisfies the following inequality

$$\| f(x+y) - f(x) - f(y) \| \leq \epsilon$$

for all $\epsilon \geq 0$ and for all $x, y \in X$, then the limit

$$a(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)$$

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exists for each $x \in X$ and $a : X \to Y$ is the unique additive function such that

$$\| f(x) - a(x) \| \leq \epsilon$$

for any $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear.

Hyers's theorem was generalized in various directions. In particular, Th. M. Rassias [7] considered a generalized version of the theorem of Hyers which permitted the Cauchy difference to become unbounded. He proved the following theorem by using a direct method: if a function $f : X \to Y$ satisfies the following inequality

$$\| f(x+y) - f(x) - f(y) \| \leq \theta(\| x \|^p + \| x \|^p)$$

for some $\theta \geq 0$, $0 \leq p < 1$, and for all $x, y \in X$, then there exists a unique additive function such that

$$\| f(x) - a(x) \| \leq \frac{2\theta}{2 - 2^p} \| x \|^p$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear. Th.M. Rassias result provided a remarkable generalization of Hyers Theorem, a fact which rekindled interest in the study of stability of functional equations. Taking this fact into consideration the Hyers-Ulam stability is called Hyers-Ulam-Rassias stability.During the last twodecades several results for the Hyers-Ulam-Rassias stability of functional equations have been proved by several mathematicians worlwide. Găvruta [4] provided a generalization of the Theorem of Th.M.Rassias.

The quadratic function $f(x) = cx^2$ $(c \in \mathbb{R})$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1}$$

Hence this question is called the quadratic functional equation, and every solution of the quadratic equation (1) is called a quadratic function.

A Hyers-Ulam stability theorem for the quadratic functional equation (1) was proved by Skof for functions $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an abelian group. In [3], Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Several functional equations have been investigated; see [8], [9], and [10].

In this paper, we will investigate the generalized Hyers-Ulam-Rassias stability of a mixed $(n, n-1)$-dimensional quadratic functional equation as follows:

$$(n-2)f(\sum_{j=1}^{n} x_j) + \sum_{i=1}^{n} f(x_i) = \sum_{1 \leq i_1 < \cdots < i_{n-1} \leq n} f(x_{i_1} + \cdots + x_{i_{n-1}}), \tag{2}$$

where $n \geq 3$ is a integer.

Before proceeding the proof, we may remarks as follows: First of all, the equation (2) can be viewed as the generalization of [[6] Equation (1.2)]. Secondly, there is a mixed $(n,2)-$dimensional quadratic functional equation: see [1].

2. A Mixed $(n,n-1)$-dimensional quadratic mapping

Lemma 2.1. *Let $n \geq 3$ be an integer, and let X, Y be vector spaces. The even mapping $f : X \to Y$ defined by*

$$(n-2)f(\sum_{j=1}^{n} x_j) + \sum_{i=1}^{n} f(x_i) = \sum_{1\leq i_1<\cdots<i_{n-1}\leq n} f(x_{i_1} + \cdots + x_{i_{n-1}})\,, \tag{3}$$

for all $x_1, \cdots, x_n \in X$. Then f is quadratic.

Proof. By letting $x_1 = \cdots = x_n = 0$ in (3), we have $(n-2)f(0) = 0$. Since $n \geq 3$, $f(0) = 0$. Also, letting $x_1 = x$, $x_2 = y$, $x_3 = -y$, and $x_k = 0\,(4 \leq k \leq n)$ in(3), we get

$$(n-2)f(x) + f(x) + 2f(y) = f(x+y) + f(x-y) + (n-3)f(x)\,.$$

Hence we may conclude that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)\,.$$

Thus f is quadratic. $\square$

3. Stability of a mixed $(n,n-1)$-dimensional quadratic mapping

Throughout in this section, let X be a normed vector space with norm $\|\cdot\|$ and Y be a Banach space with norm $\|\cdot\|$. Let $n \geq 2$ be even.

For the given mapping $f : X \to Y$, we define

$$\begin{aligned} &Df(x_1, \cdots, x_n) := \\ &\quad (n-2)f(\sum_{j=1}^{n} x_j) \quad + \quad \sum_{i=1}^{n} f(x_i) \quad - \sum_{1\leq i_1<\cdots<i_{n-1}\leq n} f(x_{i_1} + \cdots + x_{i_{n-1}})\,, \end{aligned}$$

for all $x_1, \cdots, x_n \in X$.

We will consider two cases where n is odd and $n \geq 4$ is any integer.

Theorem 3.1. *Let $n \geq 3$ be odd and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0,\infty)$ such that*

$$\tilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 4^{-j}\phi(2^j x_1, \cdots, 2^j x_n) < \infty\,, \tag{4}$$

$$\| Df(x_1, \cdots, x_n) \| \leq \phi(x_1, \cdots, x_n), \tag{5}$$

for all $x_1, \cdots, x_n \in X$. *Then there exists a unique n-dimensional quadratic mapping* $Q : X \to Y$ *such that*

$$\| f(x) - Q(x) \| \leq \frac{1}{2(n-1)} \tilde{\phi}(x, -x, x, -x, \cdots, -x, x), \tag{6}$$

for all $x \in X$.

Proof. For each $k = 1, \cdots, n$, $x_k = (-1)^{k-1}x$ in (5), we have

$$\| 2(n-1)f(x) - \frac{n-1}{2} f(2x) \| \leq \phi(x, -x, x, -x, \cdots, -x, x),$$

for all $x \in X$. Then we write

$$\| f(x) - \frac{1}{4} f(2x) \| \leq \frac{1}{2(n-1)} \phi(x, -x, x, -x, \cdots, -x, x), \tag{7}$$

for all $x \in X$.

For a given positive integer r, assume

$$\begin{aligned} \| f(x) \quad &- \quad (\frac{1}{4})^r f(2^r x) \| \\ &\leq \frac{1}{2(n-1)} \sum_{k=0}^{r-1} (\frac{1}{4})^k \phi(2^k x, -2^k x, \cdots, -2^k x, 2^k x), \end{aligned}$$

for all $x \in X$. Then if x is replaced by $2x$ in above equation, we have

$$\begin{aligned} \| f(2x) \quad &- \quad (\frac{1}{4})^r f(2^{r+1} x) \| \\ &\leq \frac{1}{2(n-1)} \sum_{k=1}^{r} (\frac{1}{4})^k \phi(2^k x, -2^k x, \cdots, -2^k x, 2^k x), \end{aligned}$$

for all $x \in X$. Now, combine (7) and the previous equation, we may conclude that

$$\begin{aligned} \| f(x) \quad &- \quad (\frac{1}{4})^t f(2^t x) \| \\ &\leq \frac{1}{2(n-1)} \sum_{k=0}^{t-1} (\frac{1}{4})^k \phi(2^k x, -2^k x, \cdots, -2^k x, 2^k x), \end{aligned}$$

for all $x \in X$ and all positive integer t. Also, letting $x = 2^r x$ in (7), we get

$$\| f(2^r x) - \frac{1}{4} f(2^{r+1} x) \| \leq \frac{1}{2(n-1)} \phi(2^r x, -2^r x, \cdots, -2^r x, 2^r x),$$

or, by multiplying $(\frac{1}{4})^r$,

$$\begin{aligned} \| (\frac{1}{4})^r f(2^r x) \quad - \quad & (\frac{1}{4})^{r+1} f(2^{r+1} x) \| \\ & \leq \frac{1}{2(n-1)} (\frac{1}{4})^r \phi(2^r x, -2^r x, \cdots, -2^r x, 2^r x), \end{aligned}$$

for all $x \in X$ and all positive integer r. This implies that for all integers $r > t > 0$,

$$(*) \qquad \begin{aligned} \| (\frac{1}{4})^r f(2^r x) \quad - \quad & (\frac{1}{4})^t f(2^t x) \| \\ & \leq \frac{1}{2(n-1)} \sum_{k=t}^{r-1} (\frac{1}{4})^k \phi(2^k x, -2^k x, \cdots, -2^k x, 2^k x), \end{aligned}$$

for all $x \in X$. Hence the sequence $\{(\frac{1}{2})^{2s} f(2^s x)\}$ is a Cauchy sequence in a Banach space Y. Hence we may define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{r \to \infty} 2^{-2r} f(2^r x),$$

for all $x \in X$. By the definition of $DQ(x_1, \cdots, x_n)$ and (5),

$$\begin{aligned} \| DQ(x_1, \cdots, x_n) \| &= \lim_{r \to \infty} (\frac{1}{2})^{2r} \| Df(2^r x_1, \cdots, 2^r x_n) \| \\ &\leq \lim_{r \to \infty} (\frac{1}{2})^{2r} \phi(2^r x_1, \cdots, 2^r x_n) = 0, \end{aligned}$$

for all $x_1, \cdots, x_n \in X$. That is, $DQ(x_1, \cdots, x_n) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. Also, letting $t = 0$ and passing the limit $r \to \infty$ in (*), we get the equation (6).

Note that

$$\begin{aligned} Q(2^j x) &= \lim_{r \to \infty} 2^{-2r} f(2^r (2^j x)) \\ &= 2^{2j} \lim_{r \to \infty} 2^{-2(r+j)} f(2^{r+j} x) \\ &= 2^{2j} Q(x). \end{aligned}$$

Now, let $Q' : X \to Y$ be another n-dimensional quadratic mapping satisfying (6). Then by previous note, we have

$$\begin{aligned} \| Q(x) - Q'(x) \| &= 2^{-2r} \| Q(2^r x) - Q'(2^r x) \| \\ &\leq \frac{2^{-2r}}{m-1} \phi(2^r x, \cdots, 2^r x), \end{aligned}$$

for all $x \in X$. As $r \to \infty$, we may conclude that $Q(x) = Q'(x)$, for all $x \in X$. Thus such a n-dimensional quadratic mapping $Q : X \to Y$ is unique. $\square$

Theorem 3.2. *Let $n \geq 3$ be odd and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\tilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 4^j \phi(2^{-j}x_1, \cdots, 2^{-j}x_n) < \infty\,, \tag{8}$$

$$\| Df(x_1, \cdots, x_n) \| \leq \phi(x_1, \cdots, x_n)\,, \tag{9}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leq \frac{1}{2(n-1)} \tilde{\phi}(x, -x, x, -x, \cdots, -x, x)\,, \tag{10}$$

for all $x \in X$.

Proof. If x is replaced by $\frac{1}{2}x$ (not $2x$), then it follows from the proof of Theorem 3.1. □

Corollary 3.3. *Let $p \neq 2$ and θ be positive real numbers, let $n \geq 3$ be odd, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$\| Df(x_1, \cdots, x_n) \| \leq \theta \sum_{i=1}^{n} ||x_i||^p\,,$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leq \frac{2n}{n-1} \cdot \frac{\theta}{|4 - 2^p|} ||x||^p\,,$$

for all $x \in X$.

Proof. Let

$$\phi(x_1, \cdots, x_n) = \theta \sum_{i=1}^{n} ||x_i||^p\,,$$

and then apply to Theroem 3.1 when $p < 2$, or apply to Theroem 3.2 when $p > 2$. □

Now, we may assume $n \geq 4$ is an integer.

Theorem 3.4. *Let $n \geq 4$, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\tilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 4^{-j} \phi(2^j x_1, \cdots, 2^j x_n) < \infty\,, \tag{11}$$

$$\| Df(x_1, \cdots, x_n) \| \leq \phi(x_1, \cdots, x_n)\,, \tag{12}$$

for all $x_1, \cdots, x_n \in X$. *Then there exists a unique* n*-dimensional quadratic mapping* $Q : X \to Y$ *such that*

$$\| f(x) - Q(x) \| \leq \frac{1}{4(m-1)} \tilde{\phi}(\underbrace{2x, -x, x, \cdots, x, -x}_{2m-terms}, 0, \cdots, 0) , \tag{13}$$

for all $x \in X$, *where* $4 \leq 2m \leq n$.

Proof. By letting $x_1 = 2x$, $x_k = (-1)^{k+1}x$, $(k = 2, \cdots, 2m)$, and $x_k = 0$, $(2m+1 \leq k \leq n)$ in (12), we have

$$\begin{aligned} \| (n-2)f(x) + f(2x) &+ (2m-1)f(x) \\ &- (mf(2x) + f(x) + (n-2m)f(x) \| \\ \leq \phi(\underbrace{2x, -x, x, \cdots, x, -x}_{2m-terms}, 0, &\cdots , 0) , \end{aligned}$$

for all $x \in X$. Then we have

$$\| f(x) - \frac{1}{4} f(2x) \| \leq \frac{1}{4(m-1)} \phi(\underbrace{2x, -x, x, \cdots, x, -x}_{2m-terms}, 0, \cdots, 0) , \tag{14}$$

for all $x \in X$.

For a given positive integer r , assume

$$\begin{aligned} &\| f(x) - (\frac{1}{4})^r f(2^r x) \| \\ \leq \quad &\frac{1}{4(m-1)} \sum_{k=0}^{r-1} (\frac{1}{4})^k \phi(\underbrace{2^{k+1}x, -2^k x, 2^k x, \cdots, 2^k x, -2^k x}_{2m-terms}, 0, \cdots, 0) , \end{aligned}$$

for all $x \in X$. Then if x is replaced by $2x$ in above equation, we have

$$\begin{aligned} &\| f(2x) - (\frac{1}{4})^r f(2^{r+1} x) \| \\ \leq \quad &\frac{1}{4(m-1)} \sum_{k=1}^{r} (\frac{1}{4})^{k-1} \phi(\underbrace{2^{k+1}x, -2^k x, 2^k x, \cdots, 2^k x, -2^k x}_{2m-terms}, 0, \cdots, 0) , \end{aligned}$$

for all $x \in X$. Now, combine (14) and the previous equation, we may conclude that

$$\begin{aligned} &\| f(x) - (\frac{1}{4})^t f(2^t x) \| \\ \leq \quad &\frac{1}{4(m-1)} \sum_{k=0}^{t-1} (\frac{1}{4})^k \phi(\underbrace{2^{k+1}x, -2^k x, 2^k x, \cdots, 2^k x, -2^k x}_{2m-terms}, 0, \cdots, 0) , \end{aligned}$$

for all $x \in X$ and all positive integer t. Also, letting $x = 2^r x$ in (14), we get

$$\begin{aligned} & \| f(2^r x) - \frac{1}{4} f(2^{r+1}x) \| \\ \leq \; & \frac{1}{4(m-1)} \phi(\underbrace{2^{r+1}x, -2^r x, 2^r x, \cdots, 2^r x, -2^r x}_{2m-terms}, 0, \cdots, 0)\,, \end{aligned}$$

or, by multiplying $(\frac{1}{4})^r$

$$\begin{aligned} & \| (\frac{1}{4})^{r+1} f(2^{r+1}x) - (\frac{1}{4})^r f(2^r x) \| \\ \leq \; & (\frac{1}{4})^r \frac{1}{4(m-1)} \phi(\underbrace{2^{r+1}x, -2^r x, 2^r x, \cdots, 2^r x, -2^r x}_{2m-terms}, 0, \cdots, 0)\,, \end{aligned}$$

for all $x \in X$ and all positive integer r. This implies that for all integers $r > t > 0$,

$$\begin{aligned} (*) \qquad & \| (\frac{1}{4})^r f(2^r x) - (\frac{1}{4})^t f(2^t x) \| \\ \leq \; & \frac{1}{4(m-1)} \sum_{k=t}^{r-1} (\frac{1}{4})^k \phi(\underbrace{2^{k+1}x, -2^k x, 2^k x, \cdots, 2^k x, -2^k x}_{2m-terms}, 0, \cdots, 0)\,, \end{aligned}$$

for all $x \in X$. Hence the sequence $\{(\frac{1}{2})^{2s} f(2^s x)\}$ is a Cauchy sequence in a Banach space Y. Hence we may define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{r \to \infty} 2^{-2r} f(2^r x)\,,$$

for all $x \in X$. By the definition of $DQ(x_1, \cdots, x_n)$ and (12),

$$\begin{aligned} \| DQ(x_1, \cdots, x_n) \| &= \lim_{r \to \infty} (\frac{1}{2})^{2r} \| Df(2^r x_1, \cdots, 2^r x_n) \| \\ &\leq \lim_{r \to \infty} (\frac{1}{2})^{2r} \phi(2^r x_1, \cdots, 2^r x_n) = 0\,, \end{aligned}$$

for all $x_1, \cdots, x_n \in X$. That is, $DQ(x_1, \cdots, x_n) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. Also, letting $t = 0$ and passing the limit $r \to \infty$ in (*), we get the equation (13).

Note that

$$\begin{aligned} Q(2^j x) &= \lim_{r \to \infty} 2^{-2r} f(2^r (2^j x)) \\ &= 2^{2j} \lim_{r \to \infty} 2^{-2(r+j)} f(2^{r+j} x) \\ &= 2^{2j} Q(x)\,. \end{aligned}$$

Now, let $Q' : X \to Y$ be another n-dimensional quadratic mapping satisfying (13). Then by previous note, we have

$$\begin{aligned} \| Q(x) - Q'(x) \| &= 2^{-2r} \| Q(2^r x) - Q'(2^r x) \| \\ &\leq \frac{2 \cdot 2^{-2r}}{4(m-1)} \phi(2^r x, \cdots, 2^r x), \end{aligned}$$

for all $x \in X$. As $r \to \infty$, we may conclude that $Q(x) = Q'(x)$, for all $x \in X$. Thus such a n-dimensional quadratic mapping $Q : X \to Y$ is unique. □

Theorem 3.5. *Let $n \geq 4$, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\tilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 4^j \phi(2^{-j} x_1, \cdots, 2^{-j} x_n) < \infty, \tag{15}$$

$$\| Df(x_1, \cdots, x_n) \| \leq \phi(x_1, \cdots, x_n), \tag{16}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leq \frac{1}{4(m-1)} \tilde{\phi}(\underbrace{2x, -x, x, \cdots, x, -x}_{2m-terms}, 0, \cdots, 0), \tag{17}$$

for all $x \in X$, where $4 \leq 2m \leq n$.

Proof. If x is replaced by $\frac{1}{2}x$ (not $2x$), then it follows from the proof of Theorem 3.4. □

Corollary 3.6. *Let $p \neq 2$ and θ be positive real numbers, let $n \geq 4$, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$\| Df(x_1, \cdots, x_n) \| \leq \theta \sum_{i=1}^{n} ||x_i||^p,$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leq \frac{n}{m-1} \cdot \frac{\theta}{|4 - 2^p|} ||x||^p,$$

for all $x \in X$, where $4 \leq 2m \leq n$.

Proof. Let

$$\phi(x_1, \cdots, x_n) = \theta \sum_{i=1}^{n} ||x_i||^p,$$

and then apply to Theroem 3.4 when $p < 2$, or apply to Theroem 3.5 when $p > 2$. □

Note that Theorem 3.4, Theorem 3.5 and Corollary 3.6 remain valid if $n \geq 4$ is either odd or even.

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