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NOTE ON THE HYERS-ULAM-RASSIAS STABILITY OF THE FIRST ORDER LINEAR DIFFERENTIAL EQUATION

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Dedicated to the Memory of Professor Donald H. Hyers

Abstract. Let X be a complex Banach space and I an open interval. We prove the stability result in the sense of Hyers-Ulam-Rassias of the X-valued differential equation

$$y'(t) + p(t)y(t) + q(t) = 0.$$

If $f\colon I \to X$ is an approximate solution of $y'+py+q = 0$, then to each $s \in I$ there corresponds an exact solution $g_s\colon I \to X$ of the differential equation above such that g_s is near to f.

1. Introduction

It seems that the stability problem of functional equations was first studied by Hyers, which was raised by Ulam (cf. [18, Chapter VI]) in 1940: "For what metric groups G is it true that an ε-automorphism of G is necessarily near to a strict automorphism?" In 1941, Hyers [4] gave an answer to this problem as follows: Let E_1, E_2 be two real Banach spaces and $f\colon E_1 \to E_2$ be a mapping.

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If there exists an $\varepsilon \geq 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in E_1$, then there exists a unique additive mapping $T\colon E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq \varepsilon$ for every $x \in E_1$. This result is called the *Hyers-Ulam stability* of the additive Cauchy equation $g(x+y) = g(x) + g(y)$.

In 1978, Rassias [11] introduced a new functional inequality that we call Cauchy-Rassias inequality and succeeded to extend the result of Hyers' by weakening the condition for the Cauchy difference to be unbounded: If there exist an $\varepsilon \geq 0$ and $0 \leq p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$, then there exists a unique additive mapping $T\colon E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\ \|x\|^p$$

for every $x \in E_1$. The stability phenomenon of this kind is called the *Hyers-Ulam-Rassias stability*. In 1991, Gajda [2] solved the problem for $1 < p$, which was raised by Rassias. In fact, the result of Rassias is valid for $1 < p$; Moreover, Gajda gave an example that a similar stability result does not hold for $p = 1$. Another example was given by Rassias and Šemrl [15, Theorem 2].

It seems that Alsina and Ger [1] are the first who consider the Hyers-Ulam stability of differential equations. They remarked that the Hyers-Ulam stability of the differential equation $y' = y$ holds: If $\varepsilon \geq 0$, f is a differentiable function on an open interval I into $\mathbb{R}$, the real number field, with $|f'(t) - f(t)| \leq \varepsilon$ for all $t \in I$, then there exists a differentiable function $g\colon I \to \mathbb{R}$ such that $g'(t) = g(t)$ and $|f(t) - g(t)| \leq 3\varepsilon$ for all $t \in I$. Many authors generalize the result of Alsina and Ger (cf. [3, 8, 9, 10, 16, 17]). Miura, Jung and Takahasi [8] proved the (generalized) Hyers-Ulam-Rassias stability of the Banach space valued differential equation $y'(t) = \lambda y(t)$ under an additional condition, where λ is a complex number. In this paper, we prove the (generalized) Hyers-Ulam-Rassias stability of the Banach space valued differential equation $y'(t) + p(t)y(t) + q(t) = 0$, where $p\colon I \to \mathbb{C}$, the complex number field, and $q\colon I \to X$ are both continuous mappings. To be more explicit, if $\epsilon\colon I \to [0, \infty)$ is a continuous mapping and if $f\colon I \to X$ is strongly differentiable with continuous derivative f' such that $\|f'(t) + p(t)f(t) + q(t)\| \leq \epsilon(t)$ for all $t \in I$, then to each $s \in I$ there corresponds a unique mapping $g_s\colon I \to X$ such that ${g_s}'(t) + p(t)g_s(t) + q(t) = 0$ and that

$$\|f(t) - g_s(t)\| \leq \frac{1}{|\tilde{p}_s(t)|} \left| \int_s^t |\tilde{p}_s(\tau)| \epsilon(\tau)\, d\tau \right|$$

for all $t \in I$, where $\tilde{p}_s(t) = \exp \int_s^t p(\tau)\, d\tau$.

2. Main results

From now on, X denotes a non-zero complex Banach space with the norm $\|\cdot\|$. We write $C(I,X)$ for the complex linear space of all X-valued continuous mappings on an interval $I \subset \mathbb{R}$. Recall that a mapping $f \in C(I,X)$ is called *strongly differentiable* if to each $t \in I$ there corresponds an element $f'(t) \in X$ such that

$$\lim_{s\to 0} \left\| \frac{f(t+s) - f(t)}{s} - f'(t) \right\| = 0.$$

We may regard f' as an X-valued mapping $t \mapsto f'(t)$ on I. We denote by $C^1(I,X)$ the linear subspace of all $f \in C(I,X)$ such that f is strongly differentiable and f' is continuous.

For each $s \in I$ and continuous function $p\colon I \to \mathbb{C}$, we define

$$\tilde{p}_s(t) \overset{\text{def}}{=} \exp \int_s^t p(\tau)\, d\tau \qquad (\forall t \in I). \tag{1}$$

Proposition 1. *Let $p\colon I \to \mathbb{C}$ be a continuous function, $s \in I$, $q \in C(I,X)$ and $f \in C^1(I,X)$. Each of the following conditions imply the other:*

(a) $f'(t) + p(t)f(t) + q(t) = 0$ *for every* $t \in I$.

(b) $f(t) = \dfrac{1}{\tilde{p}_s(t)} \left\{ f(s) - \displaystyle\int_s^t q(\tau)\tilde{p}_s(\tau)\, d\tau \right\}$ *for every* $t \in I$.

Proof. (a) $\Rightarrow$ (b). Suppose that $f'(t) + p(t)f(t) + q(t) = 0$ for every $t \in I$. Since ${\tilde{p}_s}'(t) = p(t)\tilde{p}_s(t)$ by (1), if we differentiate the mapping $f(t)\tilde{p}_s(t)$, then we get

$$\{f(t)\tilde{p}_s(t)\}' = \{f'(t) + p(t)f(t)\}\tilde{p}_s(t) = -q(t)\tilde{p}_s(t)$$

for every $t \in I$. This implies that

$$f(t)\tilde{p}_s(t) - f(s)\tilde{p}_s(s) = \int_s^t \{f(\tau)\tilde{p}_s(\tau)\}'\, d\tau = -\int_s^t q(\tau)\tilde{p}_s(\tau)\, d\tau$$

for every $t \in I$. Since $\tilde{p}_s(s) = 1$ by (1), we thus obtain

$$f(t) = \frac{1}{\tilde{p}_s(t)} \left\{ f(s) - \int_s^t q(\tau)\tilde{p}_s(\tau)\, d\tau \right\}$$

for every $t \in I$.

(b) $\Rightarrow$ (a). If f is of the form

$$f(t) = \frac{1}{\tilde{p}_s(t)} \left\{ f(s) - \int_s^t q(\tau)\tilde{p}_s(\tau)\, d\tau \right\}$$

for every $t \in I$, then we obtain

$$\begin{aligned} f'(t) &= \frac{1}{\tilde{p}_s{}^2(t)} \left[\left\{ f(s) - \int_s^t q(\tau)\tilde{p}_s(\tau)\,d\tau \right\}' \tilde{p}_s(t) \right. \\ &\qquad \left. - \left\{ f(s) - \int_s^t q(\tau)\tilde{p}_s(\tau)\,d\tau \right\} \tilde{p}_s{}'(t) \right] \\ &= \frac{1}{\tilde{p}_s{}^2(t)} \left[-q(t)\tilde{p}_s{}^2(t) - \left\{ f(s) - \int_s^t q(\tau)\tilde{p}_s(\tau)\,d\tau \right\} p(t)\tilde{p}_s(t) \right] \\ &= -q(t) - \frac{1}{\tilde{p}_s(t)} \left\{ f(s) - \int_s^t q(\tau)\tilde{p}_s(\tau)\,d\tau \right\} p(t) \\ &= -q(t) - p(t)f(t), \end{aligned}$$

and so we get $f'(t) + p(t)f(t) + q(t) = 0$ for every $t \in I$. □

Theorem 2. *Let $p\colon I \to \mathbb{C}$ and $\epsilon\colon I \to [0, \infty)$ be two continuous functions, and let $q \in C(I, X)$. If $f \in C^1(I, X)$ satisfies*

$$\left\| f'(t) + p(t)f(t) + q(t) \right\| \le \epsilon(t) \tag{2}$$

for all $t \in I$, then to each $s \in I$ there corresponds a unique $g_s \in C^1(I, X)$ such that $g_s{}'(t) + p(t)g_s(t) + q(t) = 0$ and that

$$\|f(t) - g_s(t)\| \le \frac{1}{|\tilde{p}_s(t)|} \left| \int_s^t |\tilde{p}_s(\tau)|\epsilon(\tau)\,d\tau \right| \tag{3}$$

for all $t \in I$.

Proof. Pick $s \in I$ arbitrarily, and put $w \overset{\text{def}}{=} f' + pf + q$. Note that $\|w(t)\| \le \epsilon(t)$ for every $t \in I$. An application of Proposition 1 to the equation $f' + pf + q - w = 0$ shows that

$$f(t) = \frac{1}{\tilde{p}_s(t)} \left\{ f(s) - \int_s^t (q(\tau) - w(\tau))\tilde{p}_s(\tau)\,d\tau \right\} \tag{4}$$

for all $t \in I$. We define $g_s\colon I \to X$ by

$$g_s(t) = \frac{1}{\tilde{p}_s(t)} \left\{ f(s) - \int_s^t q(\tau)\tilde{p}_s(\tau)\,d\tau \right\} \qquad (\forall t \in I). \tag{5}$$

Note that $g_s(s) = f(s)$ by (5). Another application of Proposition 1 yields that $g_s{}'(t) + p(t)g_s(t) + q(t) = 0$ for every $t \in I$. Moreover, it follows from (4) and (5) that

$$f(t) = g_s(t) + \frac{1}{\tilde{p}_s(t)} \int_s^t w(\tau)\tilde{p}_s(\tau)\,d\tau \qquad (\forall t \in I).$$

Since $\|w(t)\| \le \epsilon(t)$ for every $t \in I$, it follows that

$$\begin{aligned} \|f(t) - g_s(t)\| &= \left\| \frac{1}{\tilde{p}_s(t)} \int_s^t w(\tau)\tilde{p}_s(\tau)\, d\tau \right\| \\ &\le \frac{1}{|\tilde{p}_s(t)|} \left| \int_s^t \|w(\tau)\|\, |\tilde{p}_s(\tau)|\, d\tau \right| \\ &\le \frac{1}{|\tilde{p}_s(t)|} \left| \int_s^t \epsilon(\tau)|\tilde{p}_s(\tau)|\, d\tau \right| \end{aligned}$$

for all $t \in I$, which proves the inequality (3).

If $g \in C^1(I, X)$ is another function such that $g' + pg + q = 0$ and that

$$\|f(t) - g(t)\| \le \frac{1}{|\tilde{p}_s(t)|} \left| \int_s^t \epsilon(\tau)|\tilde{p}_s(\tau)|\, d\tau \right| \qquad (\forall t \in I),$$

then we obtain

$$\|g_s(t) - g(t)\| \le \frac{2}{|\tilde{p}_s(t)|} \left| \int_s^t \epsilon(\tau)|\tilde{p}_s(\tau)|\, d\tau \right|$$

for all $t \in I$, and hence $g_s(s) = g(s)$. Note that, by Proposition 1, g is of the form

$$g(t) = \frac{1}{\tilde{p}_s(t)} \left\{ g(s) + \int_s^t q(\tau)\tilde{p}_s(\tau)\, d\tau \right\} \qquad (\forall t \in I)$$

since $g' + pg + q = 0$. It follows from (5) with $f(s) = g_s(s) = g(s)$ that $g_s = g$, and the uniqueness is proved. □

Remark 1. The "error function"

$$\tilde{\epsilon}_s(t) \overset{\text{def}}{=} \frac{1}{|\tilde{p}_s(t)|} \left| \int_s^t \epsilon(\tau)|\tilde{p}_s(\tau)|\, d\tau \right|$$

in the right side of (3) can not be improved in general. In fact, let $p\colon I \to \mathbb{C}$ and $\epsilon\colon I \to [0, \infty)$ be continuous functions. Fix $s \in I$ and pick $x \in X$ with $\|x\| = 1$. Put

$$f_s(t) = \frac{1}{\tilde{p}_s(t)} \left\{ \int_s^t \epsilon(\tau)|\tilde{p}_s(\tau)|\, d\tau \right\} x$$

for every $t \in I$. We see that

$$\begin{aligned} {f_s}'(t) &= \frac{1}{{\tilde{p}_s}^2(t)} \left[\epsilon(t)|\tilde{p}_s(t)|\tilde{p}_s(t) - \left\{ \int_s^t \epsilon(\tau)|\tilde{p}_s(\tau)|\, d\tau \right\} p(t)\tilde{p}_s(t) \right] x \\ &= \frac{|\tilde{p}_s(t)|}{\tilde{p}_s(t)}\, \epsilon(t)x - p(t)f_s(t), \end{aligned}$$

and hence

$$\left\|{f_s}'(t) + p(t)f_s(t)\right\| = \left\| \frac{|\tilde{p}_s(t)|}{\tilde{p}_s(t)}\, \epsilon(t)x \right\| = \epsilon(t)$$

for all $t \in I$ since $\|x\| = 1$. By Theorem 2, there exists a unique $g_s \in C^1(I, X)$ such that ${g_s}'(t) + p(t)g_s(t) = 0$ and that $\|f_s(t) - g_s(t)\| \leq \tilde{\epsilon}_s(t)$ holds for all $t \in I$. Let $\theta\colon I \to [0, \infty)$ be an arbitrary function satisfying $\|f_s(t) - g_s(t)\| \leq \theta(t)$ for all $t \in I$. Since $\|x\| = 1$, we get

$$\|f_s(t)\| = \frac{1}{|\tilde{p}_s(t)|}\left|\int_s^t \epsilon(\tau)|\tilde{p}_s(\tau)|\,d\tau\right| = \tilde{\epsilon}_s(t)$$

for all $t \in I$. It follows from the uniqueness that $g_s = 0$, which implies

$$\tilde{\epsilon}_s(t) = \|f_s(t)\| = \|f_s(t) - g_s(t)\| \leq \theta(t)$$

for every $t \in I$. Therefore, the "error function" satisfying the inequality (3) can not be improved in general.

Remark 2. Miura, Jung and Takahasi [8] proved a similar result to Theorem 2 under an additional condition. Here, we give a generalized version of their result by a simple calculation: In fact, it can be proved by an application of Proposition 1.

Put $I = (a, b)$, where $-\infty \leq a < b \leq \infty$; For simplicity, we assume $0 \in I$. Let $p\colon I \to \mathbb{C}$ and $\epsilon\colon I \to [0, \infty)$ be two continuous functions, and let $q \in C(I, X)$. Suppose that $f \in C^1(I, X)$ satisfies the inequality (2) for all $t \in I$. If $\epsilon(t)|\tilde{p}_0(t)|$ and $q(t)\tilde{p}_0(t)$ are integrable on $[0, b)$, then we show that there exists a unique $g_b \in C^1(I, X)$ such that ${g_b}'(t) + p(t)g_b(t) + q(t) = 0$ and

$$\|f(t) - g_b(t)\| = \frac{1}{|\tilde{p}_0(t)|}\int_t^b \epsilon(\tau)|\tilde{p}_0(\tau)|\,d\tau$$

for every $t \in I$: The case where p is constant and $q = 0$ was proved by Miura, Jung and Takahasi [8, Theorem 1].

To prove this, put $w \overset{\text{def}}{=} f' + pf + q$. Note that $w(t)\tilde{p}_0(t)$ is integrable on $[0, b)$ since $\|w(t)\| \leq \epsilon(t)$ and since $\epsilon(t)|\tilde{p}_0(t)|$ is assumed to be integrable on $[0, b)$. By Proposition 1, with the integrability assumptions, we get

$$\begin{aligned} f(t) &= \frac{1}{\tilde{p}_0(t)}\left\{f(0) - \int_0^t (q(\tau) - w(\tau))\tilde{p}_0(\tau)\,d\tau\right\} \\ &= \frac{1}{\tilde{p}_0(t)}\left\{f(0) - \int_0^b (q(\tau) - w(\tau))\tilde{p}_0(\tau)\,d\tau \right. \\ &\qquad\qquad \left. + \int_t^b (q(\tau) - w(\tau))\tilde{p}_0(\tau)\,d\tau\right\} \end{aligned}$$

for every $t \in I$. Put

$$x_0 \overset{\text{def}}{=} f(0) - \int_0^b (q(\tau) - w(\tau))\tilde{p}_0(\tau)\,d\tau.$$

We define

$$g_b(t) \stackrel{\text{def}}{=} \frac{1}{\tilde{p}_0(t)} \left\{ x_0 + \int_t^b q(\tau)\tilde{p}_0(\tau)\, d\tau \right\}$$

for $t \in I$. By a simple calculation, we see that $g_b \in C^1(I, X)$ satisfying ${g_b}'(t) + p(t)g_b(t) + q(t) = 0$ for all $t \in I$. We now obtain

$$\begin{aligned} \|f(t) - g_b(t)\| &= \left\| -\frac{1}{\tilde{p}_0(t)} \int_t^b w(\tau)\tilde{p}_0(\tau)\, d\tau \right\| \\ &\le \frac{1}{|\tilde{p}_0(t)|} \int_t^b \|w(\tau)\|\, |\tilde{p}_0(\tau)|\, d\tau \\ &\le \frac{1}{|\tilde{p}_0(t)|} \int_t^b \epsilon(\tau)|\tilde{p}_0(\tau)|\, d\tau \end{aligned}$$

for every $t \in I$. Finally, we show the uniqueness: Suppose that $g \in C^1(I, X)$ satisfies $g'(t) + p(t)g(t) + q(t) = 0$ and

$$\|f(t) - g(t)\| \le \frac{1}{|\tilde{p}_0(t)|} \int_t^b \epsilon(\tau)|\tilde{p}_0(\tau)|\, d\tau \qquad (\forall t \in I).$$

Since g_b and g are solutions of the differential equation $y'(t)+p(t)y(t)+q(t) = 0$, it follows from Proposition 1 that

$$g_b(t) - g(t) = \frac{1}{\tilde{p}_0(t)}(g_b(0) - g(0)) \tag{6}$$

for every $t \in I$. By the triangle inequality, we get

$$\begin{aligned} \|g_b(0) - g(0)\| &= |\tilde{p}_0(t)|\, \|g_b(t) - g(t)\| \\ &\le |\tilde{p}_0(t)|(\|g_b(t) - f(t)\| + \|f(t) - g(t)\|) \\ &\le 2\int_t^b \epsilon(\tau)|\tilde{p}_0(\tau)|\, d\tau \\ &\to 0 \qquad \text{as} \qquad t \nearrow b, \end{aligned}$$

which implies $g_b(0) = g(0)$. By (6), we obtain $g_b(t) = g(t)$ for all $t \in I$, and so the uniqueness is proved.

By an argument similar to the above, we can also prove that if $\epsilon(t)|\tilde{p}_0(t)|$ and $q(t)\tilde{p}_0(t)$ are integrable on $(a, 0]$, then there exists a unique $g_a \in C^1(I, X)$ such that ${g_a}'(t) + p(t)g_a(t) + q(t) = 0$ and $\|f(t) - g_a(t)\| \le \int_a^t \epsilon(\tau)|\tilde{p}_0(\tau)|\, d\tau / |\tilde{p}_0(t)|$ for every $t \in I$.

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