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A NOTE ON COMPLEMENTARITY PROBLEMS FOR MULTIVALUED MONOTONE OPERATORS IN BANACH SPACES

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Abstract. In this paper, the existence theorems of solutions of the complementarity problems for multivalued monotone operators are proved in Banach spaces.

1. Introduction and Preliminaries

Let E be a real Banach space, E^* denotes the dual space of E, 2^{E^*} denotes the family of all nonempty subsets of E^* and $\langle \cdot, \cdot \rangle$ denotes the pairing between E^* and E. Let $K \subset E$ be a convex cone, K^* denotes the conjugate cone of K, i.e.,

$$K^* = \{u \in E^* : \langle u, x \rangle \geq 0, \forall\, x \in K\}.$$

Let $T : K \to 2^{E^*}$ be a multivalued operator, the so-called the complementarity problem of T is to find points $\bar{x} \in K$ and $\bar{u} \in T\bar{x}$ such that

$$T\bar{x} \subset K^* \text{ and } \langle \bar{u}, \bar{x} \rangle = 0.$$

The complementarity problems for multivalued non-monotone operators were discussed in [2] and the following result was proved.

Theorem A. *Let E be a real Banach space and $K \subset E$ be a closed convex cone. Suppose that $T : K \to 2^{E^*}$ is upper semicontinuous from the norm*

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topology in K to the norm topology in E^* and each Tx is norm compact; If there exist two nonempty compact subsets Q and Ω in K, for each $x \in K \setminus Q$ there exists $y \in \Omega$ such that $\inf_{u \in Tx} \langle u, x-y \rangle > 0$ and for each fixed $x \in Q$ we have

$$\inf_{u \in Tx} \langle u, y-x \rangle \geq 0 \text{ for all } y \in K.$$

Then there exist $\bar{x} \in Q \subset K$ and $\bar{u} \in T\bar{x}$ such that

$$T\bar{x} \subset K^* \text{ and } \langle \bar{u}, \bar{x} \rangle = 0.$$

The purpose of this paper is to prove the existence theorems of solutions of the complementarity problems for multivalued monotone operators in Banach spaces and to give a new method of proof different from that of [2].

We need the following lemma for the main theorems.

Lemma 1.1.[6] *Let E be a locally convex Hausdorff topological vector space and X be a nonempty compact convex subset of E. Let $T : X \to 2^{E^*}$ be monotone such that for each $x \in X$, Tx is a nonempty subset of E^* and T is lower semicontinuous from the relative topology of X to the strong topology of E^*. Then there exists a point $\hat{y} \in X$ such that*

$$\sup_{w \in T\hat{y}} Re\langle w, \hat{y}-x \rangle \leq 0 \text{ for all } x \in X.$$

We note that every Banach space is a locally convex Hausdorff topological vector space with respect to the weak topology. Therefore we have the following.

Corollary 1.2. *Let E be a Banach space and X be a nonempty weakly compact convex subset of E. Let $T : X \to 2^{E^*}$ be monotone such that for each $x \in X$, Tx is nonempty subset of E^* and T is lower semicontinuous from the weak topology of X to the norm topology of E^*. Then there exists a point $\hat{y} \in X$ such that*

$$\sup_{w \in T\hat{y}} Re\langle w, \hat{y}-x \rangle \leq 0 \text{ for all } x \in X.$$

2. Complementarity problems for monotone operators

Let E be a real Banach space, we denote by $\|\cdot\|$ the norm, by Ω° and $\partial\Omega$ the interior and the boundary of a subset Ω of E, respectively.

Theorem 2.1. *Let E be a real Banach space and $K \subset E$ be a closed convex cone. Suppose that $T : K \to 2^{E^*}$ is monotone such that for each $x \in K$, Tx is a nonempty subset of E^* and T is lower semicontinuous from the weak* *topology of K to the norm topology of E^*. If there exists a weakly compact convex subset Ω of K with $\Omega^\circ \neq \emptyset$ such that for each $x \in \Omega$, Tx is weakly* compact subset of E^* and for each $z \in \partial\Omega$, there exists $y_0 \in \Omega^\circ$ such that $\inf_{w\in Tz}\langle w, z - y_0\rangle \geq 0$. Then there exist $\bar{x} \in \Omega \subset K$ and $\bar{w} \in T\bar{x}$ such that*

$$T\bar{x} \subset K^* \text{ and } \langle \bar{w}, \bar{x}\rangle = 0.$$

Proof. First we prove that there exists $\bar{x} \in \Omega \subset K$ such that

$$\sup_{w\in T\bar{x}} \langle w, \bar{x} - y\rangle \leq 0 \text{ for all } y \in K. \tag{2.1}$$

In fact, by Corollary 1.2 there exists $\bar{x} \in \Omega$ such that

$$\sup_{w\in T\bar{x}} \langle w, \bar{x} - x\rangle \leq 0 \text{ for all } x \in \Omega. \tag{2.2}$$

If $\bar{x} \in \Omega^\circ$, then for each $y \in K$, we can choose $\lambda : 0 < \lambda < 1$ small enough so that $x = \lambda y + (1-\lambda)\bar{x} \in \Omega$. It follows from (2.2) that

$$\lambda \cdot \sup_{w\in T\bar{x}} \langle w, \bar{x} - y\rangle = \sup_{w\in T\bar{x}} \langle w, \bar{x} - x\rangle \leq 0.$$

Consequently, we have

$$\sup_{w\in T\bar{x}} \langle w, \bar{x} - y\rangle \leq 0 \text{ for all } y \in K.$$

If $\bar{x} \in \partial\Omega$, by the condition of Theorem 2.1, there exists $y_0 \in \Omega^\circ$ such that $\inf_{w\in T\bar{x}}\langle w, \bar{x} - y_0\rangle \geq 0$. By (2.2), for each $x \in \Omega$ we have

$$\langle w, \bar{x} - x\rangle \leq \langle w, \bar{x} - y_0\rangle \text{ for all } w \in T\bar{x}.$$

This implies that

$$\sup_{w\in T\bar{x}} \langle w, y_0 - x\rangle \leq 0 \text{ for all } x \in \Omega. \tag{2.3}$$

Since $y_0 \in \Omega^\circ$, for each $y \in K$, we can choose $\lambda : 0 < \lambda < 1$ small enough so that $\hat{x} = \lambda y + (1-\lambda)y_0 \in \Omega$. It follows from (2.3) that

$$\lambda \cdot \sup_{w\in T\bar{x}} \langle w, y_0 - y\rangle = \sup_{w\in T\bar{x}} \langle w, y_0 - \hat{x}\rangle \leq 0.$$

This shows that

$$\sup_{w\in T\bar{x}} \langle w, y_0 - y\rangle \leq 0 \text{ for all } y \in K. \tag{2.4}$$

Note that $y_0 \in \Omega$, by (2.2) we obtain

$$\sup_{w\in T\bar{x}} \langle w, \bar{x} - y_0\rangle \leq 0. \tag{2.5}$$

Combining (2.4) and (2.5), for all $y \in K$ we have

$$\sup_{w\in T\bar{x}} \langle w, \bar{x} - y\rangle \leq \sup_{w\in T\bar{x}} \langle w, \bar{x} - y_0\rangle + \sup_{w\in T\bar{x}} \langle w, y_0 - y\rangle \leq 0.$$

This shows that (2.1) holds.

Next we prove that the conclusion of Theorem 2.1 holds. By (2.1) we have

$$\inf_{w\in T\bar{x}}\langle w, \bar{x}-y\rangle \leq 0 \text{ for all } y\in K. \tag{2.6}$$

and

$$\inf_{w\in T\bar{x}}\langle w, y-\bar{x}\rangle \geq 0 \text{ for all } y\in K. \tag{2.7}$$

We denote by θ the zero vector of E, then $\theta\in K$, it follows from (2.6) that

$$\inf_{w\in T\bar{x}}\langle w, \bar{x}\rangle = \inf_{w\in T\bar{x}}\langle w, \bar{x}-\theta\rangle \leq 0. \tag{2.8}$$

On the other hand, since K is convex cone and $\bar{x}\in K$, so $2\bar{x}\in K$ and, by(2.7) we have

$$\inf_{w\in T\bar{x}}\langle w, \bar{x}\rangle = \inf_{w\in T\bar{x}}\langle w, 2\bar{x}-\bar{x}\rangle \geq 0. \tag{2.9}$$

Combining (2.8) and (2.9), we have $\inf_{w\in T\bar{x}}\langle w, \bar{x}\rangle = 0$. Note that the real valued function $w\mapsto \langle w, \bar{x}\rangle$ is weakly* continuous on the weakly* compact set $T\bar{x}$, so there exists $\bar{w}\in T\bar{x}$ such that

$$\langle \bar{w}, \bar{x}\rangle = \inf_{w\in T\bar{x}}\langle w, \bar{x}\rangle = 0.$$

Finally we prove that $T\bar{x}\subset P^*$. In fact, for any $w\in T\bar{x}$ and $y\in K$, by (2.7) we have

$$\langle w, y\rangle \geq \inf_{w\in T\bar{x}}\langle w, y\rangle = \inf_{w\in T\bar{x}}\langle w, y\rangle - \inf_{w\in T\bar{x}}\langle w, \bar{x}\rangle \geq \inf_{w\in T\bar{x}}\langle w, y-\bar{x}\rangle \geq 0.$$

This completes the proof. □

Remark 2.2. Theorem 2.1 generalizes Theorem 1 of Guo and Qu [5] and Theorem 2 and Theorem 3 of Zhang and Li [9] to multivalued operators, and improves Theorem 1 of Guo [1].

Remark 2.3. By the monotonicity of T, we know that the condition $\inf_{w\in Tz}\langle w, z-y_0\rangle \geq 0$ in Theorem 2.1 is replaced by $\sup_{u\in Ty_0}\langle u, z-y_0\rangle \geq 0$ and the conclusion follows.

Remark 2.4. If E is a real reflexive Banach space in Theorem 2.1, then the weakly compact convex subset Ω of E can be replaced by a relatively weak condition.

Theorem 2.5. *Let E be a real reflexive Banach space and $K\subset E$ be a closed convex cone. Suppose that $T: K\to 2^{E^*}$ is monotone such that for each $x\in K$, Tx is a nonempty subset of E^* and T is lower semicontinuous from the weak topology of K to the norm topology of E^*. If there exist a point $x_0\in K$ and a constant $\beta>0$ such that for each $x\in K$, as $\|x-x_0\|\leq\beta$, Tx is weakly* *compact and as* $\|x-x_0\|=\beta$, $\inf_{w\in Tx}\langle w, x-x_0\rangle \geq 0$. *Then there exist* $\bar{x}\in K$ *with* $\|\bar{x}-x_0\|\leq\beta$ *and* $\bar{w}\in T\bar{x}$ *such that*

$$T\bar{x}\subset K^* \text{ and } \langle \bar{w},\bar{x}\rangle = 0.$$

Proof. Setting $\Omega=\{x\in K : \|x-x_0\|\leq\beta\}$, it is easy to prove that Ω is a bounded closed convex subset in reflexive Banach space, so Ω is a weakly compact convex in K and $\Omega^\circ=\{x\in K : \|x-x_0\|<\beta\}\neq\emptyset$. For each $x\in\partial\Omega=\{x\in K : \|x-x_0\|=\beta\}$, we take $y_0=x_0\in\Omega^\circ$ and

$$\inf_{w\in Tx}\langle w, x-y_0\rangle = \inf_{w\in Tx}\langle w, x-x_0\rangle \geq 0.$$

By Theorem 2.1 and the conclusion follows. □

Especially, as x_0 is zero vector of E in Theorem 2.5 we have

Corollary 2.6. *Let* E *be a real reflexive Banach space and* $K\subset E$ *be a closed convex cone. Suppose that* $T: K\to 2^{E^*}$ *is monotone such that for each* $x\in K$, Tx *is a nonempty subset of* E^* *and* T *is lower semicontinuous from the weak topology of* K *to the norm topology of* E^*. *If there exists a constant* $\beta>0$ *and for each* $x\in K$, *as* $\|x\|\leq\beta$, Tx *is weakly compact and as* $\|x\|=\beta$, $\inf_{w\in Tx}\langle w,x\rangle\geq 0$. *Then there exist* $\bar{x}\in K$ *with* $\|\bar{x}\|\leq\beta$ *and* $\bar{w}\in T\bar{x}$ *such that*

$$T\bar{x}\subset K^* \text{ and } \langle \bar{w},\bar{x}\rangle = 0.$$

References

[1] W. P. Guo, *Complementarity problems for multivalued monotone operator in Banach spaces*, J. Math. Anal. Appl. **292** (2004), 344–350.

[2] W. P. Guo, *Complementarity problems for multivalued non-monotone operators in Banach spaces*, J. Math. Res. Exposition. **27** (2007).

[3] W. P. Guo, *Two results for the implicit complementarity problems*, J. Math. Res. Exposition. **19** (1999), 554–556.

[4] W. P. Guo, *Implicit complementarity problems for multivalued monotone operators*, Acta Analysis Functionalis Applicata. **5** (2003), 271–275.

[5] W. P. Guo and L. X. Qu, *Complementarity problems in reflexive Banach spaces*, J. Math. Study. **31** (1998), 390–393.

[6] M. H. Shih and K. K. Tan, *Generalized quasi-variational inequalities in locally convex topological vector spaces*, J. Math. Anal. Appl. **108** (1985), 333–343.

[7] W. Takahashi, *Nonlinear variational inequalities and fixed point theorems*, J. Math. Soc. Japan. **28** (1976), 168–181.

[8] M. Thera, *Existence results for the nonlinear complementarity problem and applications to nonlinear analisis*, J. Math. Anal. Appl. **154** (1991), 572–584.

[9] S. S Zhang and J. Li, *Complementarity problems in Banach spaces*, Appl. Math. J. Chinese Univ. **9** (1994), 75–83.