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VARIATIONAL INEQUALITIES AND COMPLEMENTARITY PROBLEM FOR PM-MAPS

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Abstract. Let X be a reflexive Banach space and X^* its dual space. Applying the property of pseudo–monotone map and maximal monotonicity of subdifferential, we discuss the existence of solutions of the variational inequalities and complementarity problem for PM-map in X. When both X and X^* are uniformly convex Banach spaces, We give the necessary and sufficient condition that the map T having nearest point in $J(D)$, and study the equivalent conditions to $J - T$ having zero point in D.

1. Introduction

The theory of variational inequalities were introduced by Lions, Browder, Stampacchia, Ky-Fan, has made important progress, and it is a border subject including abundant contents. Many mathematicians have paid much attention to it in theory and applications. Some nonlinear problems arising in applications have led to the study of variational inequalities for maps of the form $J - T$, that is, to find $y \in D$ such that

$$\langle Jy - Ty, y - x \rangle \leq 0, \forall x \in K,$$

where $T : D \subset K \to X^*$ is a suitable map, K is a closed convex subset of a Banach space X, X^* is the dual space of X, and $J : X \to X^*$ is the duality

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map. For example, Lan and Webb [4] study the variational inequality for maps of the form $J-T$ in [1, 4], where T is demi-continuous PM-map.

In this paper, we discuss the complementarity problem, that is, to find $y \in K$, such that

$$\langle Ay, y\rangle + \Phi(y) = 0,$$

and

$$\langle Ay - f, x\rangle + \Phi(x) \geq 0, \forall x \in K.$$

As we all know, the complementary problem has closely relation with variational inequality, with the increasing development of the variational inequality, hence, the theory and applications of complementarity problem have made progress. At last, we'll discuss the sufficient and necessary condition that the map T having nearest point in $J(D)$ equivalent to $J-T$ having zero point in D, when X and X^* are locally uniformly convex Banach spaces.

In the following, we'll list some basic concept. Let X be a Banach space and X^* its dual space.

An operator $T: X \to X^*$ is called monotone, if $\forall x, y \in K$,

$$\langle Tx - Ty, x - y\rangle \geq 0.$$

An operator $T: X \to X^*$ is called maximal monotone if it is monotone and its graph is not properly contained in the graph of any other monotone.

An operator $T: K \to X^*$ is called pseudo-monotone (see [4,5]) if for each sequence $\{y_n | n \in N\}$, such that $y_n \rightharpoonup y \in K$, and $\limsup_{n\to\infty}\langle Ty_n, y_n - y\rangle \leq 0$, then

$$\langle Ty, y - x\rangle \leq \liminf_{n\to\infty}\langle Ty_n, y_n - x\rangle, \forall x \in K.$$

An operator $T: K \to X^*$ is called S_+ type if for each sequence $\{y_n\}$, such that $y_n \rightharpoonup y \in K$ and $\limsup_{n\to\infty}\langle Ty_n, y_n - y\rangle \leq 0$, then $y_n \to y$ ([4,5]).

An operator $T: D \subset K \to X$ is called a generalized inward map (relative K) if $d(Tx, K) < \|x - Tx\|$, $x \in D, Tx \notin K$, where $d(Tx, K) = \inf\{\|Tx - z\| : z \in K\}$.

Let K be a closed convex set of a Banach space X. Then K is called a wedge if $\lambda x \in K$,$\forall x \in K$, and $\lambda \geq 0$. A wedge K is a cone if also $\{K\} \cap \{-K\} = \{0\}$ [1].

Let X be a Banach space, we always assume that X^* is strictly convex. Recall that a continuous function $\phi : [0, \infty) \to [0, \infty)$ is said to be a gauge function, if ϕ is a strictly increasing function with $\phi(0) = 0$,

$$\lim_{t\to+\infty} \phi(t) = +\infty.$$

A map $J: X \to X^*$ is said to be a duality map with gauge function ϕ [3], if for each $x \in X$,

$$\langle J(x), x\rangle = \phi(\|x\|)\|x\| \text{ and } \|Jx\| = \phi(\|x\|).$$

As we all know, if X^* is a strictly convex Banach space, the duality map in X is single-valued and demi-continuous.

Recall that a Banach space X has property (H), if $y_n \rightharpoonup y$ and $\|y_n\| \to \|y\|$ imply $y_n \to y$ [6]. It is known that a locally uniformly convex Banach space has the property (H).

Definition 1.1. A map $T : K \to X^*$ is said to be a PM-map if $J - T$ is pseudo-monotone.

Definition 1.2. A map $T : K \to X^*$ is said to be S-contractive if $J - T$ is of S_+ type.

Now, we have the important lemmas.

Lemma 1.1. [2] *Let X be a uniformly convex Banach space. Assume that $T : K \to X^*$ is an S-contractive map and $G : K \to X^*$ is compact. Then $T+G$ is S-contractive. In particular, a compact map is S- contractive. Furthermore if T is demi-continuous, then $T + G$ is a PM-map.*

Lemma 1.2. *Let X be a locally uniformly convex Banach space, $K \subset X$ be a closed convex subset, and $C : K \to X^*$ be a compact map. Then C is a PM-map.*

Proof. We only prove that $J - C$ is a pseudo-monotone map. Let $y_n \rightharpoonup y$ and $\limsup_{n\to\infty}\langle (J-C)y_n, y_n-y\rangle \le 0$. Since J is a S_+ type and $\lim_{n\to\infty}\langle Cy_n, y_n-y\rangle = 0$, we have $\lim_{n\to\infty}\langle Jy_n, y_n - y\rangle \le 0$. Therefore, $y_n \to y$. Thus for all $v \in K$

$$\begin{aligned}\liminf_{n\to\infty}\langle (J-C)y_n, y_n - v\rangle &= \liminf_{n\to\infty}\{\langle (J-C)y_n, y_n - y\rangle + \langle (J-C)y_n, y - v\rangle\}\\ &= \liminf_{n\to\infty}\langle (J-C)y_n, y - v\rangle\\ &= \langle (J-C)y, y - v\rangle,\end{aligned}$$

this implies that $J - C$ is a pseudo-monotone map. □

Lemma 1.3. [4] *Let C be a nonempty bounded closed and convex subset of a reflexive Banach space X, $T : C \to X^*$ be a bounded, demi-continuous and pseudo-monotone operator on C, and $A : C \to X^*$ be a maximal monotone operator. Let $f \in X^*$. Then there exists an element $\overline{u} \in C \cap D(A)$ such that*

$$\langle Ax + T\overline{u} - f, x - \overline{u}\rangle \ge 0, \ \ \forall x \in C.$$

Lemma 1.4. [7] *Let $p > 1$, $r > 0$ be two fixed real numbers and X be a Banach space. Then the following conditions are equivalent.*

(i) *X is uniformly convex,*

(ii) *There is a continuous, strictly increasing and convex function* $g : R^+ \to R^+$, $g(0) = 0$, *such that*

$$\|x + y\|^p \geq \|x\|^p + p\langle y, f_x\rangle + g(\|y\|),$$

for every $x, y \in B_r = \{x : \|x\| \leq r\}$, $f_x \in J_p(x) = \{x^* \in X^*, \langle x, x^*\rangle = \|x\|^p$ *and* $\|x^*\| = \|x\|^{p-1}\}$.

Lemma 1.5. [7] *Let* X *be a uniformly smooth Banach space (means* X^* *is uniformly convex). Then there exists a continuous increasing function* $b : R^+ \to R^+$ *such that* $b(0) = 0$, $b(ct) \leq cb(t)$ $(c > 1)$ *and*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|).$$

2. Complementarity problem for PM-maps

Theorem 2.1. *Let* X *be a reflexive Banach space,* $K \subset X$ *be a wedge,* $T : \overline{K}_r \to X^*$ *be a bounded, demi-continuous PM-map, (where* $\overline{K}_r = \{x \in K, \|x\| \leq r\}$ *and* $\partial K_r = \{x \in K, \|x\| = r\}$*) and* $\langle Jx - f, x\rangle + \Phi(x) \geq \langle Tx, x\rangle$, $\forall x \in \partial K_r$, *where* $\Phi(x) : X \to R$ *is proper convex lower semi-continuous function, and* $\Phi(0) = 0$, $\Phi(\lambda y) \leq \lambda\Phi(y)$ $(\forall\lambda > 0)$. *Then there exists* $y_0 \in \overline{K}_r$ *such that*

$$\langle Jy_0 - Ty_0 - f, y_0\rangle + \Phi(y_0) = 0$$

and

$$\langle Jy_0 - Ty_0 - f, x\rangle + \Phi(x) \geq 0, \quad \forall x \in K.$$

Proof. We may assume $A = J - T$, By Lemma 1.3 and maximal monotonicity of subdifferential, there exists $y_0 \in \overline{K}_r$ such that

$$\langle \partial\Phi(y_0) + Ay_0 - f, x - y_0\rangle \geq 0, \quad \forall x \in \overline{K}_r.$$

By the definition of subdifferential, we have

$$\langle Ay_0 - f, x - y_0\rangle + \Phi(x) - \Phi(y_0) \geq 0, \ \ \forall x \in \overline{K}_r.$$

In the following, we prove $\langle Ay_0 - f, y_0\rangle + \Phi(y_0) = 0$. In fact, Since $0 \in \overline{K}_r$, we have $\langle Ay_0 - f, y_0\rangle + \Phi(y_0) \leq 0$.

On the other hand, we have to prove that

$$\langle Ay_0 - f, y_0\rangle + \Phi(y_0) \geq 0.$$

If $\|y_0\| = r$, by the given boundary condition, we have $\langle Ay_0 - f, y_0\rangle + \Phi(y_0) \geq 0$. If $\|y_0\| < r$, let $x = \beta y_0$ $(\beta > 1)$ such that $x \in \overline{K}_r$, then

$$\begin{aligned} 0 &\leq \langle Ay_0 - f, \beta y_0 - y_0\rangle + \Phi(\beta y_0) - \Phi(y_0) \\ &\leq (\beta - 1)\langle Ay_0 - f, y_0\rangle + (\beta - 1)\Phi(y_0) \\ &= (\beta - 1)[\langle Ay_0 - f, y_0\rangle + \Phi(y_0)]. \end{aligned}$$

Since $\beta > 1$, $\langle Ay_0 - f, y_0\rangle + \Phi(y_0) \geq 0$.

In a word, if $\|y_0\| \leq r$, then $\langle Ay_0 - f, y_0\rangle + \Phi(y_0) = 0$, it show that

$$\langle Ay_0 - f, x\rangle + \Phi(x) \geq 0, \forall x \in \overline{K}_r.$$

This implies $\langle Ay_0 - f, x\rangle + \Phi(x) \geq 0, \forall x \in K$, since K is a wedge. □

Remark 2.1. Above Theorem generalizes the Theorem 3.2 in [4], here boundary condition $\langle Jx, x\rangle \geq \langle Tx, x\rangle$ changes into $\langle Jx - f, x\rangle \geq (Tx, x)$, $\forall x \in \partial K_r$.

Remark 2.2. If $\Phi(\lambda y) \leq \lambda^\alpha \Phi(y)$ $(\alpha > 0, \lambda > 1)$, the conclusion of Theorem 2.1 is also true. For example, if $\Phi(u) = \dfrac{1}{2}\displaystyle\int |\nabla u|^2 dx$, then $\partial\phi = -\triangle$ satisfies

$$\Phi(\lambda u) \leq \lambda^2 \Phi(u).$$

As a special case of Theorem 2.1, if $\Phi(x) \equiv 0$, Theorem 2.1 becomes following:

Corollary 2.1. [4] *Let K be a wedge in a reflexive Banach space X and let $r > 0$. Assume that $T : \overline{K}_r \to X^*$ is a bounded and demi-continuous PM-map such that, for $f \in X^*$,*

$$\langle Jx - f, x\rangle \geq \langle Tx, x\rangle, \quad \forall x \in \partial K_r.$$

Then there exists $y_0 \in \overline{K}_r$ such that

$$\langle Jy_0 - Ty_0 - f, y_0\rangle = 0$$

and

$$\langle Jy_0 - Ty_0 - f, x\rangle \geq 0, \quad \forall x \in K.$$

In the following, we study the situation that T may not be defined on $\overline{K}_r$:

Theorem 2.2. *Let X be a reflexive Banach space, $K \subset X$ be a wedge and $T : K \to X^*$ be a bounded and demi-continuous PM-map. Assume that following condition holds*

$$\lim_{\substack{x\in K\\ \|x\|\to\infty}} \frac{\langle Jx - Tx, x\rangle}{\|x\|} = \infty. \tag{P}$$

Let $\Phi(x) : X \to R$ be a proper convex lower semi-continuous function such that $\Phi(0) = 0$, and $\Phi(\lambda y) \leq \lambda\Phi(y)$ $(\lambda > 0)$. Then for every $f \in X^*$ there exists $y \in K$ such that

$$\langle Jy - Ty - f, y\rangle + \Phi(y) = 0$$

and

$$\langle Jy - Ty - f, x\rangle + \Phi(x) \geq 0, \quad \forall x \in K.$$

Proof. Since $\Phi(x)$ is a proper convex lower semi-continuous function, by the result of Barbu [2], we have $\Phi(x)$ is bounded from an affine function, that is, there exists a functional $x^* \in X^*$ and $\mu \in R$ such that $\Phi(x) \geq \langle x^*, x\rangle + \mu$. So

$$\Phi(x) \geq \langle x^*, x\rangle + \mu \geq \mu - \|x^*\|\|x\|.$$

Let $f \in X^*$ and $M^{'} > \|f\| + \|x^*\| + |\mu|$. Then, from the condition (P), there exists $r > 1$ such that

$$\langle Jx - Tx, x\rangle \geq M'\|x\|,$$

for all $x \in K$, $\|x\| \geq r$. Since for all $x \in K, \|x\| \geq r$.

$$\begin{aligned} \langle Jx - Tx - f, x\rangle + \Phi(x) &= \langle Jx - Tx, x\rangle - \langle f, x\rangle + \Phi(x) \\ &\geq M^{'}\|x\| - \|f\|\|x\| + \mu - \|x^*\|\|x\| \\ &= (M^{'} - \|f\| - \|x^*\|)\|x\| + \mu \\ &> |\mu|r + \mu > 0. \end{aligned}$$

Hence, if $x \in K$, $\|x\| \geq r$, then $\langle Jx - Tx - f, x\rangle + \Phi(x) > 0$, the remain part follows Theorem 2.1. □

Remark 2.3. The above Theorem generalizes the Theorem 2.1, that is, the domain $\overline{K}_r$ of operator T is changed into K.

Remark 2.4. The following conditions are equivalent to the condition (P).

(i) There exists $\lambda, \alpha, r > 0$ such that $\langle Jx, x\rangle \geq \lambda\|x\|^{1+\alpha}$ for all $x \in K$ with $\|x\| \geq r$ and $\limsup_{x\in K,\ \|x\|\to\infty} \dfrac{\langle Tx, x\rangle}{\|x\|^{1+\alpha}} < \lambda$.

(ii) There exists $\alpha > 0$ such that

$$\limsup_{x\in K,\ \|x\|\to\infty} \frac{\langle Tx, x\rangle}{\|x\|^{1+\alpha}} < \liminf_{t\to\infty} \frac{\phi(t)}{t^\alpha},$$

where ϕ is the gauge function of the dual map $J : X \to X^*$.

(iii) There exists $\alpha > 0$ such that

$$\liminf_{x\in K,\ \|x\|\to\infty} \frac{\langle Jx - Tx, x\rangle}{\|x\|^{1+\alpha}} > 0.$$

It is easy to see that (i) implies (ii), (ii) implies (iii), and (iii) implies (P).

As applications of Theorem 2.2, we discuss the existence of solutions for the following complementarity problem:

$$\langle Ay, y\rangle + \Phi(y) = 0, \quad \langle Ay, x\rangle + \Phi(x) \geq 0, \quad \forall x \in K, \qquad (*)$$

where $A = J - \lambda L + S, \lambda \in (0, \infty)$, L is a linear map, S is a nonlinear map, $\Phi : X \to R$ is a proper convex lower semi-continuous function, $\Phi(0) = 0$, and $\Phi(\mu y) \leq \mu\Phi(y)$, $(\forall \mu > 0)$.

In Hilbert spaces, the complementarity problem $(*)$ contains a mathematical model arising from the study of the postcritical equilibrium state of a thin plate resting, without friction, on a flat rigid support, where L is a self-adjoint compact linear map and S satisfies $\langle Sx, x\rangle > 0$ (see [1]).

Let

$$\frac{1}{\rho} = \sup_{\substack{x\in K \\ x\neq 0}} \frac{\langle Lx, x\rangle}{\|x\|^2}, \quad a = \liminf_{\substack{x\in K \\ \|x\|\to\infty}} \frac{\langle Sx, x\rangle}{\|x\|^2},$$

$$b = \liminf_{t\to\infty} \frac{\phi(t)}{t},$$

where ϕ is the gauge function of dual map $J : X \to X^*$.

Theorem 2.3. *Let X be a uniformly convex Banach space, $K \subset X$ be a wedge, $L : X \to X^*$ be a linear compact map and $\rho > 0$. If $S : K \to X^*$ is a bounded and demi-continuous PM-map, and $a > -b$, $\Phi(x) : X \to R$ is a proper convex lower semi-continuous function, $\Phi(0) = 0$, and $\Phi(\mu y) \leq \mu\Phi(y)$ $(\mu > 0)$, then for each $\lambda \in \rho[0, \rho(a + b))$, there exists $y \in K$ such that*

$$\langle Ay, y\rangle + \Phi(y) = 0$$

and

$$\langle Ay, x\rangle + \Phi(x) \geq 0, \quad \forall x \in K.$$

Proof. Since L is a linear compact map, $J - \lambda L$ is a pseudo-monotone map by Lemma 1.2. However, the sum of two pseudo-monotone maps is also pseudo-monotone, so $\lambda L - S$ is a bounded and demi-continuous PM-map. Let $\lambda \in [0, \rho(a + b))$,

$$\begin{aligned}
\limsup_{\substack{x\in K \\ \|x\|\to\infty}} \frac{\langle \lambda Lx - Sx, x\rangle}{\|x\|^2} \qquad &\leq \limsup_{\substack{x\in K \\ \|x\|\to\infty}} \frac{\langle \lambda Lx, x\rangle}{\|x\|^2} + \limsup_{\substack{x\in K \\ \|x\|\to\infty}} \frac{\langle -Sx, x\rangle}{\|x\|^2} \\
&= \limsup_{\substack{x\in K \\ \|x\|\to\infty}} \frac{\langle \lambda Lx, x\rangle}{\|x\|^2} - \liminf_{\substack{x\in K \\ \|x\|\to\infty}} \frac{\langle Sx, x\rangle}{\|x\|^2} \\
&\leq \frac{\lambda}{\rho} - a < b.
\end{aligned}$$

So for each $\lambda \in [0, \rho(a + b))$, $\lambda L + S$ satisfied the condition (iii) in Remark 2.4 with $\alpha = 1$. Hence, the result follows from Theorem 2.2. □

Remark 2.5. If $a = \infty$, then Theorem 2.3 holds for all $\lambda \in [0, \infty)$.

Corollary 2.2. *In Theorem 2.3, let $J : X \to X^*$ be a duality map, and $a > -b$ be replaced by the K-copositive condition: there exists $m \in (0, 1)$ such that*

$$\langle A_\lambda x - A_\lambda 0, x\rangle \geq m\|x\|^2,$$

for all $x \in K$ and $\lambda \geq 0$, where $A_\lambda = J - \lambda L + S$.

Let $\Phi(x) : X \to R$ *be a proper convex lower semi-continuous function,* $\Phi(0) = 0$, *and* $\Phi(\mu y) \leq \mu\Phi(y)$ $(\mu > 0)$. *Then, there exists* $y \in K$ *such that*

$$\langle Ay, y\rangle + \Phi(y) = 0,$$

and

$$\langle Ay, x\rangle + \Phi(x) \geq 0, \ \ \forall x \in K.$$

Proof. For each $\lambda \geq 0$, let $T_\lambda = \lambda L - S$. By the condition K-copositive, we have

$$\langle Jx, x\rangle \geq \langle T_\lambda x, x\rangle - \langle T_\lambda 0, x\rangle + m\|x\|^2, \forall x \in K,$$

so

$$\limsup_{\substack{x\in K \\ \|x\|\to\infty}} \frac{\langle T_\lambda x, x\rangle}{\|x\|^2} < \liminf_{t\to\infty} \frac{\phi(t)}{t} = 1.$$

Hence, by the condition of Remark 2.4 (ii) and Theorem 2.3, we can get the conclusion. □

Remark 2.6. Both the Theorem 2.3 and Corollary 2.2 generalized Theroem 3.3 and Corollary of [4]. In hear, we change conditions $y \in K$, $\langle Ay, y\rangle = 0$ and $\langle Ay, x\rangle \geq 0$, $\forall x \in K$ into $y \in K$, $\langle Ay, y\rangle + \Phi(y) = 0$ and $\langle Ay, x\rangle + \Phi(x) \geq 0$, $\forall x \in K$.

3. The discussion about zero point for map $J - T$

Theorem 3.1. *Let* X *and* X^* *be uniform convex Banach spaces,* $K \subset X$ *be a closed convex subset and* $T : D \subset K \to X^*$ *be a map. Then the following two conditions are equivalent:*

(1) *There exists* $y \in D$, *such that* $\langle Jy - Jx, J^{-1}(Jy - Ty)\rangle \leq 0$ $\forall x \in K$.

(2) T *has nearest point in* $J(D)$; *that is,* $y \in D$ *such that*

$$\|Jy - Ty\| = d(Ty, J(K)),$$

where $d(Ty, K) = inf\{\|Ty - z\| : z \in K\}$.

If also T *is generalized inward (relative* $J(K)$*), then (1) and (2) are equivalent to*

(3) $J-T$ *has zero point in* D, *that is, there exists* $y \in D$ *such that* $(J-T)y = 0$.

Proof. (1) $\Longrightarrow$ (2). Assume that (1) holds. Note that duality map is demicontinuous and one-to-one corresponding between X and X^* from the uniform convexity of X and X^*. By Lemma 1.4, we have

$$\begin{aligned} \|Ty - Jx\|^p &\geq \|Jy - Ty\|^p + \|Jy - Jx\|^p - p\langle Jy - Jx, J^{-1}(Jy - Ty)\rangle \\ &\geq \|Jy - Ty\|^p, \quad \forall x \in K. \end{aligned}$$

So, there exists $y \in D \subset K$ such that $\|Jy - Ty\| = d(Ty, J(K))$, that is, (2) holds .

(2) $\Longrightarrow$ (1). Since (2) holds, $\forall x \in K$, $J(K)$ is closed convex, by Lemma 1.5, we have

$$\begin{aligned} \|Jy - Ty\|^2 &\leq \|Ty - (1-t)Jy - tJx\|^2 \\ &\leq \|Jy - Ty\|^2 - 2t\langle Jy - Jx, J^{-1}(Jy - Ty)\rangle \\ &\quad + \max\{\|Jy - Jx\|, 1\} \cdot t\|Jy - Jx\| \cdot b(t\|Jy - Jx\|). \end{aligned}$$

Thus

$$2\langle Jy - Jx, J^{-1}(Jy - Ty)\rangle \leq \max\{\|Jy - Jx\|, 1\}\|Jy - Jx\| \cdot b(t\|Jy - Jx\|).$$

Letting $t \to 0^+$. Then, we have $b(0) = 0$ from the continuity of $b(t)$ and we obtain

$$\langle Jy - Jx, J^{-1}(Jy - Ty)\rangle \leq 0,$$

this implies that (1) holds.

If T is also an inward map, then we have

$$\|Jy - Ty\| = d(Ty, J(K)) < d(Ty, J(D)).$$

However, if T is also generalized inward, then

$$d(Ty, J(K)) \leq \|Ty - Jy\|,$$

so $Ty \in J(K)$. Hence $d(Ty, J(K)) = 0$, that is, $Jy - Ty = 0$, y is the zero point of $J - T$. $\square$

Remark 3.1. Theorem 3.1 is a generalization of Proposition 2.1 in [4] for the Banach space setting.

Corollary 3.1. *Let X and X^* be a uniformly convex Banach spaces, $K \subset X$ be a closed convex subset,and $T : K \to X^*$ be a generalized inward PM-map. Then $J - T$ is demiclosed; that is, $y_n \rightharpoonup y$ and $(J - T)y_n \to 0$, then $(J - T)y = 0$.*

Proof. Let $y_n \rightharpoonup y$ and $(J - T)y_n \to 0$. Then for every $x \in K$,

$$\lim_{n\to\infty} \langle Jy_n - Ty_n, y_n - y\rangle = 0, \quad \lim_{n\to\infty} \langle Jy_n - Ty_n, y_n - x\rangle = 0.$$

Since $J - T$ is pseudo-monotone, we have

$$\langle Jy - Ty, y - x\rangle \leq \liminf_{n\to\infty} \langle Jy_n - Ty_n, y_n - x\rangle = 0, \qquad \forall x \in K.$$

Since T is generalized inward, it follows from Theorem 3.1 that $(J - T)y = 0$. $\square$

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