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ON THE STABILITY OF ITERATIVE APPROXIMATIONS OF INVERSE-STRONGLY MONOTONE MAPPING

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Abstract. In this paper, we study iterative approximations for finding a common element of the fixed points of a nonexpansive mapping and the set of solutions of the variational inequalities for an inverse-strongly monotone mappings in Hilbert spaces. The conditons which guarantee strong convergence and stability of these approximations with respect to perturbations of nonexpansive operator S, metric projection operator P_Ω and constraint set Ω are considered. We show that the sequence converges strongly to a common element of two sets.

1. Introduction and Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let Ω be a closed convex subset of H and let P_Ω be the metric projection of H onto Ω . It is well-known that P_Ω is a nonexpansive mapping of H onto Ω and satisfies

$$\langle x - y, P_\Omega x - P_\Omega y \rangle \geq \|P_\Omega x - P_\Omega y\|^2,$$

for every $x, y \in H$. Moreover, P_Ω is characterized by the properties: $P_\Omega x \in \Omega$ and

$$\langle x - P_\Omega x, y - P_\Omega x \rangle \leq 0,$$

for all $y \in \Omega$.

Recall a mapping T of Ω into H is called monotone if for all $x, y \in \Omega$

$$\langle x - y, Tx - Ty \rangle \geq 0.$$

The variational inequality problem is to find a $u \in \Omega$ such that

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$$\langle v - u, Tu \rangle \geq 0,$$

for all $v \in \Omega$ (see [2-4]). The set of solutions of the variational inequality is denote by $VI(\Omega, T)$. A mapping T of Ω into H is called inverse-strongly monotone if there exists a positive real number α such that

$$\langle x - y, Tx - Ty \rangle \geq \alpha \|Tx - Ty\|^2,$$

for all $x, y \in \Omega$. For such a case, T is called α- inverse-strongly monotone.

Recall also a mapping T of Ω into itself is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in \Omega$. We denote by $F(T)$ the set of fixed points of T.

A mapping T of Ω into H is called strongly monotone if there exists a positive real number η such that

$$\langle x - y, Tx - Ty \rangle \geq \eta \|x - y\|^2,$$

for all $x, y \in \Omega$. In such a case, we say that T is η-strongly-monotone. If T is an α-inverse-strongly monotone mapping of Ω into H, then it is obvious that T is $\frac{1}{\alpha}$ -Lipschitz continuous. We also have that for all $x, y \in \Omega$ and $\lambda > 0$,

$$\begin{aligned}
\|(I - \lambda T)x - (I - \lambda T)y\|^2 &= \|(x - y) - \lambda(Tx - Ty)\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle x - y, Tx - Ty\rangle + \lambda^2\|Tx - Ty\|^2 \\
&\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Tx - Ty\|^2
\end{aligned}$$

so, if $\lambda \leq 2\alpha$, then $I - \lambda T$ is a nonexpansive mapping of Ω into H.

A set-valued mapping $A : H \longmapsto 2^H$ is called monotone if for all $x, y \in H, f \in Ax$ and $g \in Ay$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $A : H \longmapsto 2^H$ is maximal if the graph $G(A)$ of A is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping A is maximal if and only if for all $(x, y) \in H \times H, \langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(A)$ implies $f \in Ax$.

Let T be an inverse-strongly monotone mapping of Ω into H and let $N_\Omega v$ be the normal cone to Ω at $v \in \Omega$, i.e.,

$$N_\Omega v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in \Omega\}$$

and define

$$A\eta = \begin{cases} T\eta + N_\Omega \eta, & \eta \in \Omega, \\ \phi, & \eta \notin \Omega. \end{cases}$$

Then A is maximal monotone and $0 \in A\eta$ if and only if $\eta \in VI(\Omega, T)$(see, [6,7]).

Now we introduce several lemmas for our main results in this paper.

Lemma 1.1. [1] *Let Ω_1 and Ω_2 are two convex closed sets. If $H(\Omega_1, \Omega_2) \leq \sigma$, then there exists a positive real number C such that for all $x \in H$*

$$\|P_{\Omega_1}x - P_{\Omega_2}x\| \leq C\sqrt{\sigma},$$

where

$$H(\Omega_1, \Omega_2) = \max\{\sup_{z_1\in\Omega_1} \inf_{z_2\in\Omega_2} \|z_1 - z_2\|,\ \sup_{z_1\in\Omega_2} \inf_{z_2\in\Omega_1} \|z_1 - z_2\|\}$$

is the Hausdroff distance between Ω_1 and Ω_2.

Lemma 1.2. [5] *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space. Then, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 = {} & \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 \\ & - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.\end{aligned}$$

Lemma 1.3. [8] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose*

$$x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n,$$

for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.4. [9] *Let α_n be a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\Sigma_{n=1}^{\infty}\gamma_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} = 0$ *or* $\Sigma_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

2. Main Results

In this section, we study the stability of iterative approximation to a common element of the fixed points of a nonexpansive mapping and the set of solutions of the variational inequalities for an inverse-strongly monotone mappings in Hilbert spaces. We assume that the following conditions hold:

(P_1): Instead of Ω, there is a sequence of convex closed sets Ω_n such that the Hausdroff distance $H(\Omega, \Omega_n) \leq \sigma_n$, where $\{\sigma_n\}$ is a sequence of positive numbers with the properties (the function $\zeta(t)$ is defined below)

$$\sigma_{n+1} \leq \sigma_n, \qquad \frac{\sigma_n}{\alpha_n} \to 0, \qquad \frac{\zeta(\sigma_n)}{\alpha_n} \to 0, \text{ as } n \to \infty.$$

(P_2): On each set Ω_n, there is a nonexpansive self-mapping $S_n : \Omega_n \longmapsto \Omega_n$ satisfying the condition: there exists the increasing positive for all $t > 0$

functions $g(t)$ and $\zeta(t)$ such that $g(0) \geq 0, \zeta(0) = 0$ and if $x \in \Omega_i, y \in \Omega_j, \|x - y\| \leq \sigma$, then

$$\|S_i x - S_j y\| \leq g(\max\{\|x\|, \|y\|\})\zeta(\sigma).$$

Theorem 2.1. *Let Ω be a closed convex subset of a real Hilbert space H and let S be a nonexpansive mapping of Ω into itself. The conditions $(P_1) - (P_2)$ are fulfilled. Let $T : H \longmapsto H$ be an α-inverse-strongly monotone mapping such that $F(S) \cap VI(\Omega, T) \neq \emptyset$. Suppose $x_0 = u \in \Omega$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S_{n+1} P_{\Omega_{n+1}}(x_n - \lambda_n T x_n),$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}$ are three sequences in $[0, 1]$, satisfying $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are chosen so that

(1) $\lambda_n \in [a, b], 0 < a < b < 2\alpha$,
(2) $\lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=1}^{\infty} \alpha_n = \infty$,
(3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(4) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$,

then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(\Omega,T)}u$.

Proof. Put $y_n = P_{\Omega_{n+1}}(x_n - \lambda_n T x_n)$ and let $x^* \in F(S) \cap VI(\Omega, T)$.

Since $I - \lambda_n T$ is nonexpansive and $x^* = P_\Omega(x^* - \lambda_n T x^*)$, we have

$$\begin{aligned}
\|y_n - x^*\| &= \|P_{\Omega_{n+1}}(x_n - \lambda_n T x_n) - P_\Omega(x^* - \lambda_n T x^*)\| \\
&\leq \|P_{\Omega_{n+1}}(x_n - \lambda_n T x_n) - P_{\Omega_{n+1}}(x^* - \lambda_n T x^*)\| \\
&\quad + \|P_{\Omega_{n+1}}(x^* - \lambda_n T x^*) - P_\Omega(x^* - \lambda_n T x^*)\| \\
&\leq \|x_n - \lambda_n T x_n - (x^* - \lambda_n T x^*)\| + C\sqrt{\sigma_{n+1}} \\
&\leq \|x_n - x^*\| + C\sqrt{\sigma_{n+1}}.
\end{aligned}$$

Now we estimate $\|S_{n+1} y_n - x^*\|$ own to $H(\Omega_{n+1}, \Omega) \leq \sigma_{n+1}$, there exists $v_{n+1} \in \Omega_{n+1}$ such that $\|v_{n+1} - x^*\| \leq \Omega_{n+1}$, thus

$$\begin{aligned}
\|S_{n+1} y_n - x^*\| &= \|S_{n+1} y_n - S x^*\| \\
&\leq \|S_{n+1} y_n - S_{n+1} v_{n+1}\| + \|S_{n+1} v_{n+1} - S x^*\| \\
&\leq \|y_n v_{n+1}\| + g(\max\{\|v_{n+1}\|, \|x^*\|\})\zeta(\sigma_{n+1}) \\
&\leq \|y_n - x^*\| + \|v_{n+1} - x^*\| + g(\max\{\|v_{n+1}\|, \|x^*\|\})\zeta(\sigma_{n+1}) \\
&\leq \|y_n - x^*\| + \sigma_{n+1} + g(\max\{\|v_{n+1}\|, \|x^*\|\})\zeta(\sigma_{n+1}).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \|\alpha_n u + \beta_n x_n + \gamma_n S_{n+1} y_n - x^*\| \\
&\le \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|S_{n+1} y_n - x^*\| \\
&\le \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| \\
&\quad + \gamma_n \{\|y_n - x^*\| + \sigma_{n+1} + g(\max\{\|v_{n+1}\|, \|x^*\|\})\zeta(\sigma_{n+1})\} \\
&\le \alpha_n \|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\
&\quad + \gamma_n \{C\sqrt{\sigma_{n+1}} + \sigma_{n+1} + g(\max\{\|v_{n+1}\|, \|x^*\|\})\zeta(\sigma_{n+1})\} \\
&\le \max\{\|u - x^*\|, \|x_0 - x^*\|\} \\
&\quad + \gamma_n \{C\sqrt{\sigma_{n+1}} + \sigma_{n+1} + g(\max\{\|v_{n+1}\|, \|x^*\|\})\zeta(\sigma_{n+1})\} \\
&\le \|u - x^*\| + \gamma_n \{C\sqrt{\sigma_{n+1}} + \sigma_{n+1} + g(\max\{\|v_{n+1}\|, \|x^*\|\})\zeta(\sigma_{n+1})\}.
\end{aligned}$$

Because $\sigma_n \to 0$, $\sqrt{\sigma_n} \to 0$, $\zeta(\sigma_n) \to 0$ as $n \to \infty$, we know that $\{x_n\}$ is bounded. Hence $\{y_n\}, \{S_{n+1}y_n\}$,$\{Tx_n\}$ are also bounded.

On the other hand, we have

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&= \|P_{\Omega_{n+2}}(x_{n+1} - \lambda_{n+1} T x_{n+1}) - P_{\Omega_{n+1}}(x_n - \lambda_n T x_n)\| \\
&\le \|P_{\Omega_{n+2}}(x_{n+1} - \lambda_{n+1} T x_{n+1}) - P_{\Omega_{n+2}}(x_n - \lambda_n T x_n)\| \\
&\quad + \|P_{\Omega_{n+2}}(x_n - \lambda_n T x_n) - P_{\Omega_{n+1}}(x_n - \lambda_n T x_n)\| \\
&\le \|x_{n+1} - \lambda_{n+1} T x_{n+1} - (x_n - \lambda_n T x_n)\| + 2C\sqrt{\sigma_{n+1}} \\
&= \|x_{n+1} - \lambda_{n+1} T x_{n+1} - (x_n - \lambda_{n+1} T x_n) + (\lambda_n - \lambda_{n+1}) T x_n\| \\
&\quad + C\sqrt{\sigma_{n+1}} \\
&\le \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \|T x_n\| + 2C\sqrt{\sigma_{n+1}}.
\end{aligned} \tag{2.1}$$

We estimate $\|S_{n+2}y_{n+1} - S_{n+1}y_n\|$ own to

$$H(\Omega_{n+1}, \Omega_{n+2}) \le H(\Omega_{n+1}, \Omega) + H(\Omega_{n+2}, \Omega) \le 2\sigma_{n+1},$$

there exists $w_{n+2} \in \Omega_{n+2}$ and $\|w_{n+2} - y_n\| \le 2\sigma_{n+1}$, therefore

$$\begin{aligned}
&\|S_{n+2} y_{n+1} - S_{n+1} y_n\| \\
&= \|S_{n+2} y_{n+1} - S_{n+2} w_{n+2}\| + \|S_{n+2} w_{n+2} - S_{n+1} y_n\| \\
&\le \|y_{n+1} - w_{n+2}\| + g(\max\{\|w_{n+2}\|, \|y_n\|\})\zeta(2\sigma_{n+1}) \\
&\le \|y_{n+1} - y_n\| + \|w_{n+2} - y_n\| + g(\max\{\|w_{n+2}\|, \|y_n\|\})\zeta(2\sigma_{n+1}) \\
&\le \|y_{n+1} - y_n\| + 2\sigma_{n+1} + g(\max\{\|w_{n+2}\|, \|y_n\|\})\zeta(2\sigma_{n+1}).
\end{aligned} \tag{2.2}$$

Let $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$. Then we obtain

$$\begin{aligned} z_{n+1} - z_n &= (\frac{\alpha_{n+1}}{1-\beta_{n+1}})u + \frac{\gamma_{n+1}}{1-\beta_{n+1}} S_{n+2}y_{n+1} - \frac{\gamma_n}{1-\beta_n} S_{n+1}y_n \\ &= (\frac{\alpha_{n+1}}{1-\beta_{n+1}})u + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (S_{n+2}y_{n+1} - S_{n+1}y_n) \\ &\quad + (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) S_{n+1}y_n, \end{aligned} \tag{2.3}$$

From (2.1)-(2.3), we obtain

$$\begin{aligned} &\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\le |\frac{\alpha_{n+1}}{1-\beta_{n+1}}|\|u\| + |\frac{\gamma_{n+1}}{1-\beta_{n+1}}|\|S_{n+2}y_{n+1} - S_{n+1}y_n\| \\ &\quad + |\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}|\|S_{n+1}y_n\| - \|x_{n+1} - x_n\| \\ &\le |\frac{\alpha_{n+1}}{1-\beta_{n+1}}|\|u\| + |\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}|\|S_{n+1}y_n\| \\ &\quad - \|x_{n+1} - x_n\| + |\frac{\gamma_{n+1}}{1-\beta_{n+1}}|\{\|y_{n+1} - y_n\| + 2\sigma_{n+1} \\ &\quad + g(\max\{\|w_{n+2}\|, \|y_n\|\})\zeta(2\sigma_{n+1})\} \\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}|\|u\| + |\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}|\|S_{n+1}y_n\| \\ &\quad + |\frac{\gamma_{n+1}}{1-\beta_{n+1}}||\lambda_{n+1} - \lambda_n|\|Tx_n\| + |\frac{\gamma_{n+1}}{1-\beta_{n+1}}|\{2C\sqrt{\sigma_{n+1}} + 2\sigma_{n+1} \\ &\quad + g(\max\{\|w_{n+2}\|, \|y_n\|\})\zeta(2\sigma_{n+1})\}, \end{aligned}$$

which implies that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

By Lemma 1.3, we obtain $\|z_n - x_n\| \to 0$ as $n \to \infty$. Consequently

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|z_n - x_n\| = 0.$$

From (2.1) we also have $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.

Since $x_{n+1} - x_n = \alpha_n(u - x_n) + \gamma_n(S_{n+1}y_n - x_n)$, $\|S_{n+1}y_n - x_n\| \to 0$ as $n \to \infty$.

Put $M = \sup\{\|v_n\|, \|w_n\|, \|y_n\|, \|x^*\|\}$. Then, by Lemma 1.4, for $x^* \in F(S) \cap VI(\Omega, T)$, we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n u + \beta_n x_n + \gamma_n S_{n+1} y_n - x^*\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|S_{n+1} y_n - x^*\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \{\|y_n - x^*\| + \sigma_{n+1} + g(M)\zeta(\sigma_{n+1})\}^2 \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|y_n - x^*\|^2 + \gamma_n \{2\|y_n - x^*\|\sigma_{n+1} \\
&\quad + \sigma_{n+1}^2 + 2(\|y_n - x^*\| + \sigma_{n+1}) g(M)\zeta(\sigma_{n+1}) + g(M)^2 \zeta(\sigma_{n+1})^2\} \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n (\|x_n - \lambda_n T x_n - (x^* - \lambda_n T x^*)\| \\
&\quad + C\sqrt{\sigma_{n+1}})^2 \\
&\quad + \gamma_n \{2\|y_n - x^*\|\sigma_{n+1} + 2(\|y_n - x^*\| + \sigma_{n+1}) g(M)\zeta(\sigma_{n+1}) \\
&\quad + \sigma_{n+1}^2 + g^2(M)\zeta^2(\sigma_{n+1})\} \\
&= \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n d_n \\
&\quad + \gamma_n \|x_n - \lambda_n T x_n - (x^* - \lambda_n T x^*)\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n d_n \\
&\quad + \gamma_n \{\|x_n - x^*\|^2 + \lambda_n(\lambda_n - 2\alpha)\|T x_n - T x^*\|^2\} \\
&\le \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 + \gamma_n d_n + \alpha_n a(b - 2\alpha)\|T x_n - T x^*\|^2,
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
&\gamma_n d_n \\
&\ = \gamma_n \{2\|y_n - x^*\|\sigma_{n+1} + \sigma_{n+1}^2 + 2(\|y_n - x^*\| + \sigma_{n+1}) g(M)\zeta(\sigma_{n+1}) \\
&\quad + g(M)^2 \zeta(\sigma_{n+1})^2\} \\
&\quad + 2\|x_n - \lambda_n T x_n - (x^* - \lambda_n T x^*)\| C\sqrt{\sigma_{n+1}} \\
&\quad + C^2 \sigma_{n+1}.
\end{aligned} \tag{2.5}$$

From (2.4) and (2.5), we have

$$\begin{aligned}
&-\alpha_n a(b - 2\alpha)\|T x_n - T x^*\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \gamma_n d_n + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + \gamma_n d_n + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad \times (\|x_n - x_{n+1}\|).
\end{aligned}$$

Since $\alpha_n \to 0$, $d_n \to 0$, $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we obtain $\|T x_n - T x^*\| \to 0$ as $n \to \infty$.

On the other hand

$$
\begin{aligned}
&\|y_n - x^*\|^2 \\
&\le \|P_{\Omega_{n+1}}(x_n - \lambda_n T x_n) - P_{\Omega_{n+1}}(x^* - \lambda_n T x^*)\|^2 + C^2\sigma_{n+1} \\
&\quad + 2\|P_{\Omega_{n+1}}(x_n - \lambda_n T x_n) - P_{\Omega_{n+1}}(x^* - \lambda_n T x^*)\|C\sqrt{\sigma_{n+1}} \\
&\le \langle (x_n - \lambda_n T x_n) - (x^* - \lambda_n T x^*), y_n - P_{\Omega_{n+1}}(x^* - \lambda_n T x^*)\rangle \\
&\quad + 2\|P_{\Omega_{n+1}}(x_n - \lambda_n T x_n) - P_{\Omega_{n+1}}(x^* - \lambda_n T x^*)\|C\sqrt{\sigma_{n+1}} \\
&\quad + C^2\sigma_{n+1} \\
&\le \langle x_n - \lambda_n T x_n - (x^* - \lambda_n T x^*), y_n - x^*\rangle \\
&\quad + |x_n - \lambda_n T x_n - (x^* - \lambda_n T x^*)|\sigma_{n+1} + C^2\sigma_{n+1} \\
&\quad + 2\|P_{\Omega_{n+1}}(x_n - \lambda_n T x_n) - P_{\Omega_{n+1}}(x^* - \lambda_n T x^*)\|C\sqrt{\sigma_{n+1}} \\
&= \langle x_n - \lambda_n T x_n - (x^* - \lambda_n T x^*), y_n - x^*\rangle + e_n \\
&\le \frac{1}{2}\|x_n - \lambda_n T x_n - (x^* - \lambda_n T x^*)\|^2 + \frac{1}{2}\|y_n - x^*\|^2 \\
&\quad - \frac{1}{2}\|x_n - \lambda_n T x_n - (x^* - \lambda_n T x^*) - y_n - x^*\|^2 + e_n \\
&\le \frac{1}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|y_n - x^*\|^2 - \frac{1}{2}\|x_n - y_n - \lambda_n(T x_n - T x^*)\|^2 + e_n \\
&= \frac{1}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|y_n - x^*\|^2 - \frac{1}{2}\|x_n - y_n\|^2 \\
&\quad + \lambda_n\langle x_n - y_n, T x_n - T x^*\rangle - \frac{1}{2}\lambda_n^2\|T x_n - T x^*\|^2 + e_n,
\end{aligned}
\tag{2.6}
$$

where

$$
\begin{aligned}
e_n = {}& 2\|P_{\Omega_{n+1}}(x_n - \lambda_n T x_n) - P_{\Omega_{n+1}}(x^* - \lambda_n T x^*)\|C\sqrt{\sigma_{n+1}} \\
&+ |x_n - \lambda_n T x_n - (x^* - \lambda_n T x^*)|\sigma_{n+1} \\
&+ C^2\sigma_{n+1}
\end{aligned}
\tag{2.7}
$$

Therefore

$$
\begin{aligned}
\|y_n - x^*\|^2 \le {}& \|x_n - x^*\|^2 + \|x_n - y_n\|^2 \\
&+ 2\lambda_n\langle x_n - y_n, T x_n - T x^*\rangle \\
&- \lambda_n^2\|T x_n - T x^*\|^2 + 2e_n.
\end{aligned}
\tag{2.8}
$$

From (2.6)-(2.8), we obtain

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|y_n - x^*\|^2 \\
&\quad + \gamma_n\{2\|y_n - x^*\|\sigma_{n+1} + \sigma_{n+1}^2 + 2(\|y_n - x^*\| + \sigma_{n+1})g(M)\zeta(\sigma_{n+1}) \\
&\quad + g^2(M)\zeta^2(\sigma_{n+1})\}
\end{aligned}
$$

$$\begin{aligned}
&\le \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2-\gamma_n\|x_n-y_n\|^2\\
&\quad+2\gamma_n\lambda_n\langle x_n-y_n,Tx_n-Tx^*\rangle-\gamma_n\lambda_n^2\|Tx_n-Tx^*\|^2+\gamma_n e_n\\
&\quad+\gamma_n\{2\|y_n-x^*\|\sigma_{n+1}+\sigma_{n+1}^2+2(\|y_n-x^*\|+\sigma_{n+1})g(M)\zeta(\sigma_{n+1})\\
&\quad+g^2(M)\zeta^2(\sigma_{n+1})\}\\
&\le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\gamma_n\|x_n-y_n\|^2\\
&\quad+2\gamma_n\lambda_n\|x_n-y_n\|\|Tx_n-Tx^*\|+h_n.
\end{aligned}\tag{2.9}$$

Put

$$\begin{aligned}
h_n=\gamma_n\{&2\|y_n-x^*\|\sigma_{n+1}+\sigma_{n+1}^2+2(\|y_n-x^*\|+\sigma_{n+1})g(M)\zeta(\sigma_{n+1})\\
&+g(M)^2\zeta(\sigma_{n+1})^2\}+\gamma_n e_n.
\end{aligned}\tag{2.10}$$

From (2.9) and (2.10), we also have

$$\begin{aligned}
\gamma_n\|x_n-y_n\|^2&\le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\quad+2\gamma_n\lambda_n\|x_n-y_n\|\|Tx_n-Tx^*\|+h_n\\
&\le \alpha_n\|u-x^*\|^2+2\gamma_n\lambda_n\|x_n-y_n\|\|Tx_n-Tx^*\|+h_n\\
&\quad+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)(\|x_n-x^*\|-\|x_{n+1}-x^*\|).
\end{aligned}$$

Since $\alpha_n\to 0$, $\|x_n-x_{n+1}\|\to 0$, $\|Tx_n-Tx^*\|\to 0$, $h_n\to 0$ as $n\to\infty$, therefore $\|x_n-y_n\|\to 0$ as $n\to\infty$.

On the other hand, since

$$\|S_{n+1}y_n-y_n\|\le\|S_{n+1}y_n-x_n\|+\|x_n-y_n\|,$$

we obtain $\|S_{n+1}y_n-y_n\|\to 0$ as $n\to\infty$. Let $z_0=P_{F(S)\cap VI(\Omega,T)}u$.

Next we show that

$$\limsup_{n\to\infty}\langle u-z_0,x_n-z_0\rangle\le 0.$$

To show this, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$, such that

$$\limsup_{n\to\infty}\langle u-z_0,S_{n+1}y_n-z_0\rangle=\lim_{n\to\infty}\langle u-z_0,S_{n_i+1}y_{n_i}-z_0\rangle.$$

As $\{y_{n_i}\}$ is bounded, we have that a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ converges weakly to z. We may assume without loss of generality that $y_{n_i}\rightharpoonup z$. Since $\|S_{n+1}y_n-y_n\|\to 0$, we obtain $S_{n_i+1}y_{n_i}\rightharpoonup z$ as $i\to\infty$. Then we can obtain $z\in F(S)\cap VI(\Omega,T)$. In fact, let us first show that $z\in VI(\Omega,T)$. Let

$$A\eta=\begin{cases}T\eta+N_\Omega\eta, & \eta\in\Omega,\\ \phi, & \eta\notin\Omega.\end{cases}$$

and

$$A_n\eta=\begin{cases}T\eta+N_{\Omega_n}\eta, & \eta\in\Omega_n,\\ \phi, & \eta\notin\Omega_n.\end{cases}$$

Let $(\eta,\xi)\in G(A)$. Since $\xi-T\eta\in N_\Omega\eta$, we have $\langle \eta-u,\xi-T\eta\rangle\geq 0,\forall u\in\Omega$.

For $\{y_n\}\in\Omega_{n+1}$, there exists $\{y'_n\}\in\Omega$, such that $\|y_n-y'_n\|\leq\sigma_{n+1}$, then

$$\langle \eta-y_n,\xi-T\eta\rangle=\langle \eta-y'_n,\xi-T\eta\rangle+\langle y'_n-y_n,\xi-T\eta\rangle$$

and

$$\begin{aligned}\langle \eta-y_n,\xi\rangle&=\langle \eta-y'_n,\xi-T\eta\rangle+\langle y'_n-y_n,\xi-T\eta\rangle+\langle \eta-y_n,T\eta\rangle\\&\geq\langle \eta-y_n,T\eta\rangle+\langle y'_n-y_n,\xi-T\eta\rangle.\end{aligned}$$

On the other hand, by the properties of P_Ω, for all $u\in\Omega_{n+1}$

$$\langle x_n-\lambda_nTx_n-y_n,u-y_n\rangle\geq 0.$$

For $\eta\in\Omega$, there exists $\eta'_n\in\Omega_{n+1}$, such that $\|\eta'_n-\eta\|\leq\sigma_{n+1}$. So we have

$$\langle x_n-\lambda_nTx_n-y_n,\eta'_n-y_n\rangle\leq 0,$$

and

$$\langle \eta'_n-y_n,\tfrac{y_n-x_n}{\lambda_n}+Tx_n\rangle\leq 0\ \ .$$

Thus

$$\begin{aligned}\langle \eta-y_{n_i},\xi\rangle&=\langle \eta-y'_{n_i},\xi-T\eta\rangle+\langle y'_{n_i}-y_{n_i},\xi-T\eta\rangle+\langle \eta-y_{n_i},T\eta\rangle\\
&\geq\langle \eta-y_{n_i},T\eta\rangle+\langle y'_{n_i}-y_{n_i},\xi-T\eta\rangle\\
&\geq\langle \eta-y_{n_i},T\eta\rangle+\langle \eta'_{n_i}-y_{n_i},\frac{y_{n_i}-x_{n_i}}{\lambda_{n_i}}+Tx_{n_i}\rangle\\
&\quad+\langle y'_{n_i}-y_{n_i},\xi-T\eta\rangle\\
&=\langle \eta-y_{n_i},T\eta\rangle-\langle \eta-y_{n_i},\frac{y_{n_i}-x_{n_i}}{\lambda_{n_i}}+Tx_{n_i}\rangle\\
&\quad-\langle \eta-\eta_{n_i},\frac{y_{n_i}-x_{n_i}}{\lambda_{n_i}}+Tx_{n_i}\rangle+\langle y'_{n_i}-y_{n_i},\xi-T\eta\rangle\\
&=\langle \eta-y_{n_i},T\eta-Tx_{n_i}\rangle-\langle \eta-\eta_{n_i},\frac{y_{n_i}-x_{n_i}}{\lambda_{n_i}}\rangle\\
&\quad-\langle \eta'_{n_i}-\eta,\frac{y_{n_i}-x_{n_i}}{\lambda_{n_i}}+Tx_{n_i}\rangle+\langle y'_{n_i}-y_{n_i},\xi-T\eta\rangle\\
&=\langle \eta-y_{n_i},T\eta-Ty_{n_i}\rangle+\langle \eta-y_{n_i},Ty_{n_i}-Tx_{n_i}\rangle\\
&\quad-\langle \eta-\eta_{n_i},\frac{y_{n_i}-x_{n_i}}{\lambda_{n_i}}\rangle-\langle \eta'_{n_i}-\eta,\frac{y_{n_i}-x_{n_i}}{\lambda_{n_i}}+Tx_{n_i}\rangle\\
&\quad+\langle y'_{n_i}-y_{n_i},\xi-T\eta\rangle\\
&\geq\langle \eta-y_{n_i},Ty_{n_i}-Tx_{n_i}\rangle-\langle \eta-\eta_{n_i},\frac{y_{n_i}-x_{n_i}}{\lambda_{n_i}}\rangle\\
&\quad-\langle \eta'_{n_i}-\eta,\frac{y_{n_i}-x_{n_i}}{\lambda_{n_i}}+Tx_{n_i}\rangle+\langle y'_{n_i}-y_{n_i},\xi-T\eta\rangle.\end{aligned}\tag{2.11}$$

By (2.11), we know that $\langle \eta - z, \xi \rangle \geq 0$ as $i \to \infty$. Since A is maximal monotone, we have $z \in A^{-1}0$, therefore $z \in VI(\Omega, T)$. Let us show that $z \in F(S)$. Suppose $z \notin F(S)$, from Opial's condition,

$$\begin{aligned}
\liminf_{i\to\infty} \|y_{n_i} - z\| &< \liminf_{i\to\infty} \|y_{n_i} - Sz\| \\
&\leq \liminf_{i\to\infty} \|y_{n_i} - S_{n_i+1}y_{n_i}\| + \liminf_{i\to\infty} \|S_{n_i+1}y_{n_i} - Sz\| \\
&\leq \liminf_{i\to\infty} \|S_{n_i+1}y_{n_i} - S_{n_i+1}v_{n_i+1}\| + \liminf_{i\to\infty} \|S_{n_i+1}v_{n_i+1} - Sz\| \\
&\leq \liminf_{i\to\infty} \|y_{n_i} - v_{n_i+1}\| + \liminf_{i\to\infty} g(M)\zeta(\sigma_{n+1}) \\
&\leq \liminf_{i\to\infty} \|y_{n_i} - z\| + \liminf_{i\to\infty} \|v_{n_i+1} - z\| \\
&= \liminf_{i\to\infty} \|y_{n_i} - z\|.
\end{aligned}$$

This is a contradiction. Thus we obtain $z \in F(S)$. Then we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle u - z_0, x_n - z_0 \rangle &= \limsup_{n\to\infty} \langle u - z_0, S_{n+1}y_n - z_0 \rangle \\
&= \lim_{n\to\infty} \langle u - z_0, S_{n_i+1}y_{n_i} - z_0 \rangle \\
&= \langle u - z_0, z - z_0 \rangle \\
&\leq 0.
\end{aligned} \tag{2.12}$$

Therefore

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \langle \alpha_n u + \beta_n x_n + \gamma_n S_{n+1}y_n - z_0, x_{n+1} - z_0 \rangle \\
&= \alpha_n \langle u - x_0, x_{n+1} - z_0 \rangle + \beta_n \langle x_n - z_0, x_{n+1} - z_0 \rangle \\
&\quad + \gamma_n \langle S_{n+1}y_n - z_0, x_{n+1} - z_0 \rangle \\
&\leq \alpha_n \langle u - x_0, x_{n+1} - z_0 \rangle + \frac{1}{2}\beta_n (\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
&\quad + \frac{1}{2}\gamma_n (\|S_{n+1}y_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
&\leq \alpha_n \langle u - x_0, x_{n+1} - z_0 \rangle + \frac{1}{2}\beta_n (\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
&\quad + \frac{1}{2}\gamma_n \{\|S_{n+1}y_n - Sy_n'\|^2 + 2\|S_{n+1}y_n - Sy_n'\|\|Sy_n' - SZ_0\| \\
&\quad + \|Sy_n' - SZ_0\|^2 + \|x_{n+1} - z_0\|^2\} \\
&\leq \alpha_n \langle u - x_0, x_{n+1} - z_0 \rangle + \frac{1}{2}\beta_n (\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
&\quad + \frac{1}{2}\gamma_n \{g^2(M)\zeta^2(\sigma_{n+1}) + 2g(M)^2\|y_n' - z_0\|\zeta(\sigma_{n+1}) \\
&\quad + \|y_n' - z_0\|^2 + \|x_{n+1} - z_0\|^2\}
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n\langle u-x_0, x_{n+1}-z_0\rangle + \frac{1}{2}\beta_n(\|x_n-z_0\|^2+\|x_{n+1}-z_0\|^2)\\
&\quad + \frac{1}{2}\gamma_n\{\|y_n-z_0\|^2+2\|y_n'-z_0\|\sigma_{n+1}+\sigma_{n+1}^2+g^2(M)\zeta^2(\sigma_{n+1})\\
&\quad + 2g(M)\|y_n'-z_0\|\zeta(\sigma_{n+1})+\|x_{n+1}-z_0\|^2\}\\
&\le \alpha_n\langle u-x_0, x_{n+1}-z_0\rangle + \frac{1}{2}\beta_n(\|x_n-z_0\|^2+\|x_{n+1}-z_0\|^2)\\
&\quad + \frac{1}{2}\gamma_n\{\|x_n-z_0\|^2+\|x_{n+1}-z_0\|^2+2\|x_n-z_0\|\sigma_{n+1}+2\sigma_{n+1}^2\\
&\quad + 2\|y_n-z_0\|\sigma_{n+1}+2g(M)\|y_n'-z_0\|\sigma_{n+1}+g^2(M)\zeta^2(\sigma_{n+1})\}.
\end{aligned}
\tag{2.13}
$$

Simplify (2.13) we obtain

$$
\begin{aligned}
\|x_{n+1}-z_0\|^2 = (1-\alpha_n)\|x_n-z_0\|^2 &+ 2\alpha_n\langle u-x_0, x_{n+1}-z_0\rangle\\
&+ \gamma_n\{2\|x_n-z_0\|\sigma_{n+1}+2\|y_n-z_0\|\sigma_{n+1}+2\sigma_{n+1}^2\\
&+ 2g(M)\|y_n'-z_0\|\sigma_{n+1}+g^2(M)\zeta^2(\sigma_{n+1})\}.
\end{aligned}
$$

From (2.12), we know that $\limsup_{n\to\infty}\langle u-x_0, x_{n+1}-z_0\rangle \le 0$, and by introduction we obtain

$$
\begin{aligned}
&\limsup_{n\to\infty}\frac{\gamma_n}{\alpha_n}\{2\|x_n-z_0\|\sigma_{n+1}+2\|y_n-z_0\|\sigma_{n+1}\\
&+2\sigma_{n+1}^2+2g(M)\|y_n'-z_0\|\sigma_{n+1}+g^2(M)\zeta^2(\sigma_{n+1})\}=0.
\end{aligned}
$$

By Lemma 1.4, we have $\|x_{n+1}-z_0\|\to 0$ as $n\to\infty$. This completes the proof. □

Corollary 2.2. *Let Ω be a closed convex subset of a real Hilbert space H, and let $T: H \longmapsto H$ be an α-inverse-strongly monotone mapping such that $F(S)\cap VI(\Omega, T)\neq\emptyset$. The condition (P_1) of this section is filled. Suppose $x_0 = u\in\Omega$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_{\Omega_{n+1}}(x_n - \lambda_n T x_n),$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying $\alpha_n+\beta_n+\gamma_n = 1$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are chosen so that

(1) $\lambda_n\in[a,b], 0<a<b<2\alpha$,
(2) $\lim_{n\to\infty}\alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n=\infty$,
(3) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$,
(4) $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n)=0$,

then $\{x_n\}$ converges strongly to $P_{VI(\Omega,T)}u$.

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