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MAPPINGS WITH UNCOUNTABLY MANY TOPOLOGICALLY CRITICAL POINTS AND APPLICATIONS TO SOME CURVATURE PROBLEMS

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Dedicated to the Memory of Professor Grigorios F. Tsagas in admiration

Abstract. In this paper we improve the results of [9, 10] showing that, under certain topological conditions, the topological φ-category of a pair (M, N) of topological manifolds is infinite uncountable. The improved result is then applied to improve the results of [4], showing that various L_{top} categories are infinite uncountable.

1. Introduction

Let M, N be topological manifolds such that $\dim M = \dim N$ and let $f : M \to N$ be a continuous mapping. We call a point $p \in M$ to be a *topological regular point* of f if f is a local homeomorphism at p. Otherwise p is a *topological critical point* of f. Denote by $R_{top}(f)$ and $C_{top}(f)$ the set of topological regular points and the set of topological critical points respectively and observe that $R_{top}(f)$ is open while $C_{top}(f)$ is closed. Another important associated set of f is the set of its *topological critical values* $B_{top}(f) = f\big(C_{top}(f)\big)$.

When M, N are differentiable manifolds and $f : M \to N$ is a differentiable mapping, the similar notions of regular and critical points of f are usually given in terms of the rank of the tangent mapping, the sets $R(f)$ and $C(f)$ of regular and critical points have the same properties above mentioned.

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Examples 1.1. 1. If $f:\mathbb{R}\to\mathbb{R}, f(x)=x^3$, then $C_{top}(f)=\emptyset\subset\{0\}=C(f)$;
2. If $f:S^2\to\mathbb{R}^2, f(x,y,z)=(x,y)$, then it can be easily checked that $C_{top}(f)=C(f)=\{(x,y,z)\in S^2\,|\,z=0\}$.

While the classical Sard's theorem for differentiable mappings ensures us that the set of critical values is of zero measure in N, so it is a small set for any differential mapping, we have been concerned, in some previous papers [3, 4, 8, 9, 10], with showing that, under certain topological conditions on the given manifolds M and N, the critical set cannot be arbitrarily small with respect to cardinality, it actually having uncountably many critical points for any mapping acting between such manifolds. In the present paper we are improving first the previous results of [9] and [10], showing that they are still valid at the topological level, and then apply the improved results to show that various L_{top} categories are infinite uncountable.

Related to the critical set we define, for a pair (M,N) of differentiable manifolds, the so called *φ-category* as

$$\varphi(M,N)=\min\{\#C(f)\mid f\in C^\infty(M,N)\}.$$

In a completely analogous way, for a pair (M,N) of topological manifolds, define the so called *topological φ-category* as

$$\varphi_{top}(M,N)=\min\{\#C_{top}(f)\mid f\in C(M,N)\}$$

and observe that for any two differentiable manifolds M,N having the same dimension the following inequality holds

$$\varphi_{top}(M,N)\le\varphi(M,N).$$

A survey on these categories is given in the paper [2] and some other details in the book [1]. Related to the φ-category of a pair of differentiable manifolds having the same dimension, we have:

Theorem 1.2. ([9]) *Let M, N be compact connected differentiable manifolds having the same dimension m. The following statements are true:*

(i) *If $m\ge 3$ and $\pi_1(M)$ cannot be embedded as a subgroup in $\pi_1(N)$, then $\varphi(M,N)=\aleph_1$;*

(ii) *If $m\ge 4$ and $\pi_q(M)\not\simeq\pi_q(N)$ for some $q\in\{2,\dots,m-2\}$, then $\varphi(M,N)=\aleph_1$.*

2. Basic Results

Theorem 2.1. ([11]) *Let M be an m-dimensional topological manifold and let A be a closed, at most countable subset of M. If P is a compact differentiable k-dimensional manifold ($k < m, \partial P \neq \emptyset$) and if $f : P \to M$ is a continuous mapping such that $f(\partial P) \subseteq M\backslash A$, then there exists a continuous mapping $g : P \to M$ such that $g(P) \subseteq M\backslash A, g\Big|_{\partial P} = f\Big|_{\partial P}$ and $f \simeq g(rel\partial P)$.*

Corollary 2.2. *If M^m is a connected topological manifold ($\partial M = \emptyset$) and if $A \subseteq M$ is a closed at most countable subset of M, then $M\backslash A$ is also connected and $\pi_q(M, M\backslash A) \simeq 0$ for all $q \in \{1, \ldots, m-1\}$. In particular, using the Hurewicz's theorem, we get that the natural group homomorphism*

$$\chi_{m-1} : \pi_{m-1}(M, M\backslash A) \to H_{m-1}(M, M\backslash A)$$

is an isomorphism. On the other hand the inclusion $i_{M\backslash A} : M\backslash A \hookrightarrow M$ is $(m-1)$-connected, that is the induced group homomorrphism

$$\pi_q(i_{M\backslash N}) : \pi_q(M\backslash N) \hookrightarrow \pi_q(M)$$

is an isomorphism for $q \leq m-2$ and it is an epimorphism for $q = m-1$.

Proof. The connectedness of $M\backslash A$ follows easily from Theorem 2.1 by considering the particular case $P = [0, 1]$. The fact that $\pi_q(M, M\backslash A) = 0$ for all $q \in \{1, 2, \ldots, m-1\}$ is an immediate consequence of Theorem 2.1 and of the fact that $[\alpha] \in \pi_q(M, M\backslash A)$ is zero if and only if there exists $\beta \in [\alpha]$ such that $\beta(D^q) \subseteq M\backslash A$. Further on, using the exact homotopy sequence

$$\cdots \to \pi_{r+1}(M, M\backslash A) \to \pi_r(M\backslash A) \overset{\pi_r(i_{M\backslash A})}{\longrightarrow} \pi_r(M) \to \pi_r(M, M\backslash A) \to \cdots,$$

and the relations $\pi_q(M, M\backslash A) = 0$ for all $q \in \{1, 2, \ldots, m-1\}$, it follows that the inclusion $i_{M\backslash A} : M\backslash A \hookrightarrow M$ is $(m-1)$-connected. □

3. The topological version of the theorem 1.2

In this section we are going to give the arguments in proving the topological version of Theorem 1.2.

Theorem 3.1. *Let M,N be compact connected topological manifolds having the same dimension $m \geq 2$. If the continuous mapping $f : M \to N$ is* *surjective and its topological critical set is at most countable, then the set $f^{-1}(B_{top}(f))$ is at most countable and its restriction*

$$M\backslash f^{-1}(B_{top}(f)) \xrightarrow{g} N\backslash B_{top}(f), p \mapsto f(p)$$

is a covering mapping with finitely many sheets.

Proof. The fact that the restriction g is a finitely sheeted covering mapping follows, for instance, from [6, Theorem 4.22,pp 29], taking into account that g is a proper local homeomorphism.

It remains only to prove that $f^{-1}(f(B_{top}(f))$ is at most countable. Obviously $B_{top}(f) = f(C_{top}(f)$ is at most countable and for any $q \in N$ the set $f^{-1}(q) = (f^{-1}(q) \cap C_{top}(f)) \cup (f^{-1}(q) \cap R_{top}(f))$ is at most countable because both $f^{-1}(q) \cap C_{top}(f)$ and $f^{-1}(q) \cap R_{top}(f)$ have this property too, being discrete. In particular the closed set $f^{-1}(f(B_{top}(f)) = \bigcup_{q \in B_{top}(f)} f^{-1}(q)$ is also at most countable, as an at most countable union of at most countable sets. □

Corollary 3.2. *Let M, N be connected topological manifolds such that* dim M $\geq$ *dim* $N \geq 2$*. If $f : M \to N$ is a non-surjective closed continuous mapping, then either $C_{top}(f) = M$, or f has an infinite uncountable number of topological critical values. Therefore, in any case, f has an infinite uncountable number of topological critical points. If M is compact and N is noncompact, then one particularly gets that $\varphi_{top}(M, N) = \aleph_1$.*

The proof of Corollary 3.2 uses Theorem 2.1 and it is completely similar with that of [10, Theorem 1.1] (see also [8, Theorem 2.1]).

Proposition 3.3. *Let M be an m-dimensional topological manifold ($m \geq 2$ and $\partial M = \emptyset$) and let A be a closed, at most countable subset of M. If M is connected, then $M\backslash A$ is also connected and the inclusion $i : M\backslash A \hookrightarrow M$ is (m-1)-connected, that is the homomorphism $\pi_q(i) : \pi_q(M\backslash A) \to \pi_q(M)$, induced by the inclusion, is an isomorphism for $q \leq m - 2$ and it is an epimorphism for $q = m - 1$.*

The proof of Proposition 3.3 uses Theorem 2.1 and it is completely similar with that of [10, Proposition 2.3].

Theorem 3.4. *Let M, N be compact connected topological manifolds having the same dimension m. The following statements are true:*

(i) *If $m \geq 3$ and $\pi_1(M)$ cannot be embedded as a subgroup in $\pi_1(N)$, then $\varphi_{top}(M, N) = \aleph_1$;*

(ii) *If* $m \geq 4$ *and* $\pi_q(M) \not\simeq \pi_q(N)$ *for some* $q \in \{2, \ldots, m-2\}$, *then* $\varphi_{top}(M, N) = \aleph_1$.

Proof. We have to show that any continuous mapping $f : M \to N$ has an infinite uncountable number of topological critical points. If f is not surjective, this follows easily by Corollary 3.2. Assume that $f : M \to N$ is surjective and that f has at most a countable number of topological critical points. Hence, by Theorem 3.1 the restriction $g = f|_{M \backslash f^{-1}(B_{top}(f))} : M \backslash f^{-1}(B_{top}(f)) \to N \backslash B_{top}(f)$ is a covering map. It follows that

$$g_1 : \pi_1(M \backslash f^{-1}(B_{top}(f))) \to \pi_1(N \backslash B_{top}(f))$$

is a monomorphism and

$$g_q : \pi_q(M \backslash f^{-1}(B_{top}(f))) \to \pi_q(N \backslash B_{top}(f))$$

are isomorphisms for all $q \geq 2$. On the other hand, by Theorem 3.1 and Proposition 3.3 it follows that the homomorphisms

$$i_q : \pi_q(M \backslash f^{-1}(B_{top}(f))) \to \pi_q(M) \quad \text{and} \quad j_q : \pi_q(N \backslash B_{top}(f)) \to \pi_q(N)$$

induced by the inclusions

$$i : M \backslash f^{-1}(B_{top}(f)) \to M \quad \text{and} \quad j : N \backslash B_{top}(f) \to N$$

are isomorphisms for all $q \in \{0, 1, \ldots, m-2\}$. From the commutative diagram

$$\begin{tikzcd} M \backslash f^{-1}(B_{top}(f)) \arrow[r, "g"] \arrow[d, "i"'] & N \backslash B_{top}(f) \arrow[d, "j"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

we get the following commutative diagrams

$$\begin{tikzcd} \pi_q(M \backslash f^{-1}(B_{top}(f))) \arrow[r, "g_q"] \arrow[d, "i_q"'] & \pi_q(N \backslash B_{top}(f)) \arrow[d, "j_q"] \\ \pi_q(M) \arrow[r, "f_q"] & \pi_q(N) \end{tikzcd}$$

(i) In the case when $q = 1$, because $f_1 \circ i_1 = j_1 \circ g_1$, i_1, j_1 are isomorphisms and g_1 is a monomorphism, it follows that $f_1 = j_1 \circ g_1 \circ i_1^{-1}$ is a monomorphism, that is a contradiction with the hypothesis of the statement (i).

(ii) For $q \in \{2, \ldots, m-2\}$, i_q, j_q, g_q are isomorphisms which together with $f_q \circ i_q = j_q \circ g_q$ leads to the conclusion that $f_q = j_q \circ g_q \circ i_q^{-1}$ are isomorphisms for all $q \in \{2, \ldots, m-2\}$, that is a contradiction with the hypothesis of the statement (ii). $\square$

Corollary 3.5. (i) *If M and N are compact topological manifolds having the same dimension $m \geq 3$ and $\#\big(\pi_1(M)\big) > \#\big(\pi_1(N)\big)$, then $\varphi_{top}(M,N) = \aleph_1$. Therefore, one particularly gets $\varphi_{top}(T^m, S^m) = \aleph_1$ and that*

$$\varphi_{top}(P^n(\mathbb{R}) \times P^k(\mathbb{R}), P^{n+k}(\mathbb{R})) = \aleph_1$$

where n, k are two natural numbers such that $n + k \geq 3$ and $P^s(\mathbb{R})$ is the real s-dimensional projective space.

(ii) *Under the same conditions on the manifolds M, N as above, if $\pi_1(M)$ and $\pi_1(N)$ are finite groups such that*

$$\Big(\#\big(\pi_1(M)\big), \#\big(\pi_1(N)\big)\Big) = 1,$$

then $\varphi_{top}(M,N) = \aleph_1$.

(iii) *If m, k are two natural numbers such that n, $k \geq 2$, then*

$$\varphi_{top}(T^{n+k}, T^n \times S^k) = \aleph_1, \ \varphi_{top}(T^n \times S^k, T^{n+k}) = \aleph_1,$$

$$\varphi_{top}(S^{n+k}, S^n \times S^k) = \aleph_1, \ \varphi_{top}(S^n \times S^k, S^{n+k}) = \aleph_1,$$

where T^p denotes the p dimensional torus $\underbrace{S^1 \times \cdots \times S^1}_{p\ times}$.

4. The G and L categories of a manifold and some geometric applications

For a differentiable m-dimensional manifold we denote by $k(M)$ the smallest natural number such that M is immersible in $\mathbb{R}^{m+k(M)}$. By means of Whitney's theorem, observe that $k(M) \leq m + 1$ and if M^m is a compact manifold, then $k(M) \leq m - 1$.

Let M^m be an orientable manifold immersible in $\mathbb{R}^{m+1}$, $f : M \to \mathbb{R}^{m+1}$ be an immersion and $N_f : M \to S^m$ its associated Gauss mapping. Because the Gauss-Kronecker curvature is defined as $K_f(p) = \det(dN_f)_p$ it follows that $K_f(p) = 0$ iff $p \in C(N_f)$, that is

$$C(N_f) = \{p \in M | K_f(p) = 0\}.$$

Therefore we can define the $G-category$ of M as

$$G(M) = \min\{\#C(N_f) | f \in Imm(M, \mathbb{R}^{m+1})\},$$

where $Imm(M, \mathbb{R}^{m+1})$ is the set of all immersions of M into $\mathbb{R}^{m+1}$.

In a completely analogous way we define the *topological G-category* of M as

$$G_{top}(M) = \min\{\#C_{top}(N_f) \mid f \in Imm(M, \mathbb{R}^{m+1})\},$$

and observe that

$$\begin{aligned} \varphi_{top}(M, S^m) &\le \min\{G_{top}(M), \varphi(M, S^m)\} \\ &\le \max\{G_{top}(M), \varphi(M, S^m)\} \le G(M). \end{aligned} \tag{1}$$

Remark 4.1. Let us consider the regular surface $(S)\, z = x^4 + y^4$. We will give the arguments for the equalities

$$C(N) = \{(x, 0, x^4) \mid x \in \mathbb{R}\} \cup \{(0, y, y^4) \mid y \in \mathbb{R}\},\ C_{top}(N) = \emptyset \tag{2}$$

where $N = N_i : S \to S^2$ is the usual Gauss mapping and $i : S \hookrightarrow \mathbb{R}^3$ is the inclusion.

Indeed, one can easily check, using the equality $K(x, y, z) = \det(dN)_{(x,y,z)}$ and a well known formula for $K(x, y, z)$, that

$$\det(dN)_{(x,y,z)} = K(x, y, z) = \frac{144x^2y^2}{\big(16(x^6 + y^6) + 1\big)^2},\ (x, y, z) \in S.$$

Because $C(N) = \{(x, y, z) \in S \mid \det(dN)_{(x,y,z)} = 0\}$, the first equality of (2) follows immediately.

On the other hand, the formula of $N : S \to S^2$ is given by

$$N(x, y, z) = \frac{1}{\sqrt{16(x^6 + y^6) + 1}}(-4x^3, -4y^3, 1),$$

meaning that the image of N is contained in the north hemisphere U of S^2. Therefore a local representation of N is $\alpha = \pi \circ N \circ r : \mathbb{R}^2 \to \mathbb{R}^2$, where $r : \mathbb{R}^2 \to S$, $r(u, v) = (u, v, u^4 + v^4)$ and $\pi : U \to \mathbb{R}^2$, $\pi(x, y, z) = (x, y)$. The formula of α is $\alpha(u, v) = \frac{1}{\sqrt{16(u^6+v^6)+1}}(-4u^3, -4v^3)$ and it gives a homeomorphism between $\mathbb{R}^2$ and its image

$$B(0, 1) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\},$$

the inverse being $\beta : B(0, 1) \to \mathbb{R}^2$,

$$\beta(x, y) = \Big(-\Big[\frac{x}{4\sqrt{1 - x^2 - y^2}}\Big]^{1/3}, -\Big[\frac{y}{4\sqrt{1 - x^2 - y^2}}\Big]^{1/3}\Big).$$

Consequently the equality $C_{top}(N) = \emptyset$ holds.

Let us also mention that the case of the $P^2(\mathbb{R})$-valued Gauss maps of a given surface immersed in all possible ways in the Minkowski 3-space and their singular (parabolic) sets has been treated in [7].

According to Theorem 3.4 and the inequalities of (1) we obtain:

Proposition 4.2. *If M is an m dimensional orientable manifold immersible in $\mathbb{R}^{m+1}$, then we have:*

(i) *If $m \geq 3$ and M is not simply connected, then $G(M) = G_{top}(M) = \aleph_1$;*

(ii) *If $m \geq 4$ and $\pi_q(M)$ is not trivial for some $q \in \{2, \dots, m-2\}$, then $G(M) = G_{top}(M) = \aleph_1$.*

Corollary 4.3. *If $k, n_1, \dots, n_k$ are natural numbers such that $k \geq 2$ and $n_1 + \cdots + n_k \geq 3$, then $S^{n_1} \times \cdots \times S^{n_k}$ is obviously orientable and immersible in $\mathbb{R}^{n_1+\cdots+n_k+1}$ and*

$$G(S^{n_1} \times \cdots \times S^{n_k}) = G_{top}(S^{n_1} \times \cdots \times S^{n_k}) = \aleph_1.$$

Therefore any immersion

$$f : S^{n_1} \times \cdots \times S^{n_k} \to \mathbb{R}^{n_1+\cdots+n_k+1}$$

has uncountably many points of zero Gauss-Kronecker curvature.

The Corollary 4.3 follows immediately from Proposition 4.2, taking into account the fact that the product $S^{n_1} \times \cdots \times S^{n_k}$ is orientable and some of its homotopy groups of are not trivial (see [9, Corollary 4.2]).

Let us also mention that the inequality $\varphi_{top}(M_g, S^2) \geq 3$ can be proved in a completely analogous way like the same inequality at differentiable level, the last one being done in [9, Theorem 4.3].

The above mentioned G-categories can be extended to the so called L_k-categories considering immersions of arbitrary high codimension instead of immersions of codimension one and replacing the Gauss-Kronecker curvature with the Lipschitz-Killing curvature of such an immersion.

Let M^m be a differentiable manifold immersible in $\mathbb{R}^{m+k}$, $Imm^k(M)$ be the set of all immersions of M into $\mathbb{R}^{m+k}$, $f \in Imm^k(M)$ be an immersion and $v \in (df)_p(T_p(M))^\perp \cap S^{m+k-1}$, where $p \in M$ is a given point. Consider the second fundamental form $\phi^v_{f,p}$ of f and the projections $N_f : \mathcal{N}^{f,1} \to S^{m+k-1}$, $\pi : \mathcal{N}^{f,1} \to M$, where

$$\begin{aligned}\mathcal{N}^{f,1} &= \{(x, v) \in M \times S^{m+k-1} \ : \ v \perp (df)_p(T_p(M))\} \\ &= \bigcup_{p \in M} \Big(\{p\} \times \big(S^{m+k-1} \cap (df)_p(T_p(M))^\perp \big) \Big).\end{aligned}$$

Observe that π is a fibration with fibre S^{k-1}. Recall that the *Lipschitz-Killing curvature* of f at the point $p \in M$ in the direction v is defined as $L_{f,v} = \frac{\det(\phi^v_{f,p})}{\det(g_f(p))}$. It is clear that $\phi^{v_0}_{f,p_0}$ is degenerated if and only if (p_0, v_0) is a critical point of N_f. Therefore

$$C(N_f) = \{(p,v) \in \mathcal{N}^{f,1} \ : \ L_{f,v}(p) = 0\},$$

that is the φ-category of the pair $(\mathcal{N}^{f,1}, S^{m+k-1})$ is a lower bound for the minimum number of points of zeros of the Lipschitz-Killing curvature with respect to all immersions of M in $\mathbb{R}^{m+k}$. This fact can be shortly written as

$$L_k(M) \geq \varphi(\mathcal{N}^{f,1}, S^{m+k-1}),$$

where $L_k(M) = \min\{\#C(N_f) \ : \ f \in Imm^k(M)\}$ is the so called L_k-*category* or the *immersiability category* of M which was defined and studied in [4].

In a completely similar way one can define the *topological* L^k-category of M as $L^k_{top}(M) = \min\{\#C_{top}(N_f) \ : \ f \in Imm^k(M)\}$, and the inequality $L_k(M) \geq L^k_{top}(M)$ is obvious. Taking into account that in this paper we argued all the preparatory results at topological level, it is also obvious that all the valid results for L_k category, at differential level, remains valid for L^k_{top} category, at topological level.

Theorem 4.4. *Let M^m be a connected closed differentiable manifold.*

(i) *If $m \geq 1$ and $\pi_1(M)$ is not trivial, then $L_k(M) = L^k_{top}(M) = \aleph_1$ for all $k \geq \max\{3, k(M)\}$;*

(ii) *If $m \geq 2$ and $\pi_q(M)$ is not trivial for some for some $q \in \{2, \ldots, k-1\}$, then $L_k(M) = L^k_{top}(M) = \aleph_1$ for all $k \geq \max\{3, k(M)\}$.*

Theorem 4.5. *Let M^m be a connected closed differentiable manifold immersible in $\mathbb{R}^{m+2}$.*

(i) *If $m \geq 2$ and $\pi_1(M)$ is not trivial, then $L_2(M) = L^2_{top}(M) = \aleph_1$;*

(ii) *If $m \geq 3$ and $\pi_2(M) \not\simeq \mathbb{Z}$, then $L_2(M) = L^2_{top}(M) = \aleph_1$;*

(iii) *If $m \geq 4$ and $\pi_q(M)$ is not trivial for some $q \in \{3, \ldots, m-1\}$, then $L_2(M) = L^2_{top}(M) = \aleph_1$.*

Examples 4.6. (i) If $m \geq 1$, then $L_k(P^m(\mathbb{R})) = L^k_{top}(P^m(\mathbb{R})) = \aleph_1$, for all $k \geq \max\{3, k(P^m(\mathbb{R}))\}$;

(ii) If $m \geq 1$, then $L_k(L^{2m-1}(q_1, \ldots, q_p)) = L^k_{top}(L^{2m-1}(q_1, \ldots, q_p)) = \aleph_1$, for all $k \geq \max\{3, k(L^{2m-1}(q_1, \ldots, q_p))\}$;

(iii) For $m \geq 2$ we have that $L_k(SO_m) = L^k_{top}(SO_m) = \aleph_1$, for all $k \geq \max\{3, k(SO_m)\}$.

(iv) $L_k(G_{p,l}) = L^k_{top}(G_{p,l}) = \aleph_1$, for all $k \geq \max\{3, k(G_{p,l})\}$;

(v) If $m \geq 3$, then $L_k(Spin_m) = L^k_{top}(Spin_m) = \aleph_1$, for all $k \geq \max\{4, k(Spin_m)\}$.

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