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APPROXIMATE SOLUTION OF FRACTIONAL BLACK-SCHOLE'S EUROPEAN OPTION PRICING EQUATION BY USING ETHPM

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Abstract. We proposed a new reliable combination of new Homotopy Perturbation Method (HPM) and Elzaki transform called as Elzaki Transform Homotopy Perturbation Method (ETHPM) is designed to obtain a exact solution to the fractional Black-Scholes equation with boundary condition for a European option pricing problem. The fractional derivative is in Caputo sense and the nonlinear terms in Fractional Black-Scholes Equation can be handled by using HPM. The Black-Scholes formula is used as a model for valuing European or American call and put options on a non-dividend paying stock. The methods give an analytic solution of the fractional Black-Scholes equation in the form of a convergent series. Finally, some examples are included to demonstrate the validity and applicability of the proposed technique.

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1. Introduction

Fractional calculus is a part of the study of mathematics and the basic concepts of fractional calculus are not new. Since many years, fractional differential equations have been investigated by many people. It is used in many fields of science and technology [1, 4, 15, 21, 22, 27, 29]. But some fractional differential equations do not have exact solutions, so we require new methods and integral transforms. Several methods and integral transforms have been designed for solving such fractional differential equations. One of such transform is known as Elzaki transform, it was initially introduced by Elzaki in 2011. Elzaki derived this transform for ordinary and partial differential equations in the time domain [13, 16, 17, 18]. But first time in 2012, Elzaki and Hilal, also developed a mixture of homotopy perturbation and Elzaki transform for solving nonlinear partial differential equations [19]. Later on many researchers used this mixture for solving linear and nonlinear partial differential equations [20] and then it is also used for solving fractional partial differential equations [9]. In 2012, Kumar et al. used the mixture of homotopy perturbation and Laplace transform for solving fractional Black-Scholes European option pricing equation [30]. Also in 2013, Elbeleze et al. proposed a new method known as Sumudu transform homotopy perturbation method for fractional Black-Scholes European option pricing equations [2]. From last few years many researchers have been paying their attention on the existence of solution of the Black Scholes model using different methods [3, 5, 6, 12, 14, 24, 25, 28, 33].

Black and Scholes in 1973 [11] got an idea that would change the world of finance forever. Black-Scholes is a pricing model used to determine the fair price or theoretical value for a call or a put option based on six variables such as volatility, type of options, underlying stock price, time, strike price, and risk-free rate. In 2000 Manale and Mahomed have modified a simple formula for valuing American and European calll and put options [31]. Black-Scholes pricing model is mostly used by the traders who buy options that are priced under the formula calculated value, and sell options that title are priced higher than the Black-Scholes calculated value. This model for pricing stock options has been applied for many different commodities and payoff structures, because it is very easy to use. The Black-Scholes model for value of an option is shown by the following equation:

$$\frac{\partial u}{\partial t}+\frac{\sigma x^2}{2}\frac{\partial^2 u}{\partial x^2}+r(t)x\frac{\partial u}{\partial x}-r(t)u, \quad 0<\mu\leq 1, \tag{1.1}$$

where $u(x,t)$, $(\mathrm{x},\mathrm{t})\in \mathrm{R}^{+}\times(0,\mathrm{T})$ is the European option price at asset price x and at time t, T is maturity, $r(t)$ is the risk-free interest rate and $\sigma(\mathrm{x},\mathrm{t})$ represents the volatility function of the underlying asset. The payoff functions are $\mathrm{v_c(x,t) = max(x-E,0)}$ denotes the European call option value, $\mathrm{v_p(x,t) = max(E-x,0)}$ denotes the European put option value, E denotes the expiration price for the option and the function $\max(x,0)$ gives the large value between x and 0.

In this article, we consider the following form of the Black-Schools fractional differential equation:

$$\frac{\partial^\mu u}{\partial t^\mu} + \frac{\sigma x^2}{2}\frac{\partial^2 u}{\partial x^2} + r(t)x\frac{\partial u}{\partial x} - r(t)u, \quad 0 < \mu \leq 1, \tag{1.2}$$

subject to the condition

$$u(x,0) = \max\left(e^x - 1, 0\right).$$

The aim of this article is to study the application of ETHPM, to obtain approximate solution of fractional Black-Schools equation with initial conditions of the form

$$D_t^\mu u = u_{xx} + (k-1)u_x - ku, \quad 0 < \mu \leq 1,$$

and the result is presented.

2. Elzaki transform

A new transform called the Elzaki transform defined for function of exponential order [7, 8, 10], we consider functions in the set A defined by

$$A = \{f(t) : \exists M, K_1, K_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, \text{if } t \in (-1)^j\,[0,\infty)\}.$$

In the set A, the constant M is finite number, $\mathrm{K_1, K_2}$ are finite or infinite. The Elzaki transform denoted by the operator $\mathrm{E}(\cdot)$ and defined by the integral equation.

$$E[f(t)] = T(v) = v\int_0^\infty f(t)e^{\frac{-t}{v}}dt, \quad t \geq 0, \quad K_1 \leq v \leq K_2, \quad 0 \leq t < \infty.$$

3. Fundamental facts of the fractional calculus

Here, we mention some basic fundamental properties of the fractional calculus.

Definition 3.1. ([23, 32]) The Riemann-Liouville fractional integral operator of order $\mu \geq 0$, of a function $f \in C_\mu, \mu \geq -1$ is defined by:

$$\begin{aligned} J^\mu f(x) &= \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1} f(t)dt, \ \mu > 0, \ x > 0, \\ J^0 f(x) &= f(x). \end{aligned}$$

Properties of the operator J^μ can be found, we mention only the followings:

(1) $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$, and $\gamma > -1$,

(2) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$ and $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,

(3) $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantage, when trying to model real world phenomenon with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_t^μ proposed by Caputo in his work.

Definition 3.2. ([26]) The fractional derivative of f(x) in the Caputo sense is defined by:

$$D_*^\mu f(x) = J^{m-\mu} D^m f(x) = \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^m(t) dt,$$

for $m-1 < \mu \leq m$, $m \in \mathbb{N}$, $x > 0$.

For the Riemann-Liouville fractional integral and the Caputo fractional derivative, we have the following relation:

$$J_T^\mu D_t^\mu f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0_+) \frac{x^k}{k!} \quad x > 0.$$

Definition 3.3. ([18]) The Elzaki transform of the Caputo fractional derivative is defined by:

$$E\left[\boldsymbol{D}_t^\mu f(t)\right] = \boldsymbol{\nu}^{-\mu} \left\{ T(\nu) - \sum_{k=1}^{n} \nu^{\mu-k+2} \left[D^{\mu-K} \left. (f(t)) \right|_{t=0} \right] \right\}.$$

4. Elzaki transform homotopy perturbation method

Consider a general nonlinear non-homogeneous partial fractional differential equation with initial condition of the form:

$$D_t^\mu u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \tag{4.1}$$

$$Lu(x,o) = h(x), \quad u_t(x,o) = f(x), \tag{4.2}$$

where $D_t^\mu u(x,t)$ is the Caputo fractional derivative of the function $u(x,t)$, R is linear differential operator, N is the general nonlinear differential operator and $g(x,t)$ is the source term.

Taking Elzaki transform on both sides of equation (4.1), to get

$$E[D_t^\mu u(x,t)] + E[Ru(x,t)] + E[Nu(x,t)] = E[g(x,t)].$$

Using the differentiation property of Elzaki transforms and initial condition (4.2), we have

$$E[u(x,t)] = f(x) + u^{\mu}E[g(x,t)] - u^{\mu}E^{-1}[E[Ru(x,t)] + E[Nu(x,t)]]. \quad (4.3)$$

Appling the inverse Elzaki transform on both sides of equation (4.3), to find

$$u(x,t) = g(x,t) - p.E^{-1}[v^{\mu}[E[Ru(x,t)] + E[Nu(x,t)]]], \quad (4.4)$$

where $g(x,t)$ represent the term arising from the source term and the prescribed initial condition.

Now we apply the HPM

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \quad (4.5)$$

and the nonlinear term can be decomposed as

$$\mathrm{N}u(x,t) = \sum_{n=0}^{\infty} p^n \mathrm{H}_n(u), \quad (4.6)$$

where $\mathrm{H_n(u)}$ are He's polynomials and given by

$$\mathrm{H}_n(u_0, u_1, u_2, \ldots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[\mathrm{N}\left(\sum_{n=0}^{\infty} p^i u^i\right)\right]_{p=0}, \; n = 0, 1, 2, \ldots.$$

Substituting (4.5) and (4.6) in (4.4), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = g(x,t) - p\left\{E^{-1}\left[v^{\mu}E\left(R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right)\right]\right\}.$$

This is the coupling of the Elzaki transform and the HPM, comparing the coefficients of like powers of the following approximations are obtained.

$$\begin{aligned}
&p^0 : u_0(x,t) = g(x,t),\\
&p^1 : u_1(x,t) = -E^{-1}\left[Ev^{\mu}\left[Ru_0(x,t) + H_0(u)\right]\right],\\
&p^2 : u_2(x,t) = -E^{-1}\left[Ev^{\mu}\left[Ru_1(x,t) + H_1(u)\right]\right],\\
&p^3 : u_3(x,t) = -E^{-1}\left[Ev^{\mu}\left[Ru_2(x,t) + H_2(u)\right]\right].
\end{aligned}$$

Proceeding in this same manner, the rest of the components $u_n(x,t)$ can be completely obtained and the series solution is thus entirely determined, finally we approximate the analytical $u(x,t)$ as

$$u(x,t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots.$$

Theorem 4.1. *The function $u(x,t)$ is defined by*

$$u(x,t) = \max\left(e^x, o\right)\left(1 - e^{-kt}\right) + \max\left(e^x - 1, o\right)e^{-kt}, \tag{4.7}$$

$$i.e.,\ u(x,t) = e^x(1 - e^{-kt}) + (e^x - 1)e^{-kt}$$

satisfies the equation

$$\frac{\partial^\mu u}{\partial t^\mu} = \frac{\partial^2 u}{\partial x^2} + (k-1)\frac{\partial u}{\partial x} - ku, \quad 0 < \mu \leq 1. \tag{4.8}$$

Proof. We have

$$\frac{\partial u}{\partial t} = ke^{-kt}, \ \ \frac{\partial u}{\partial x} = e^x, \ \ \frac{\partial^2 u}{\partial x^2} = e^x. \tag{4.9}$$

Substituting Eqs.(4.7) and (4.9) into the Eq.(4.8) gives (when $\mu = 1$)

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - (k-1)\frac{\partial u}{\partial x} + ku \\
&= ke^{-kt} - e^x - (k-1)e^x + k[e^x(1 - e^{-kt}) + (e^x - 1)e^{-kt}] \\
&= ke^{-kt} - e^x - ke^x + e^x + ke^x - ke^x e^{-kt} + ke^x e^{-kt} - ke^{-kt} \\
&= 0.
\end{aligned}$$

This completes the proof. □

Theorem 4.2. *The function $u(x,t)$ is defined by*

$$u(x,t) = x(1 - e^{-0.06t}) + \max\left(x - 25e^{-0.06}, 0\right)e^{-0.06t},$$

$$i.e.,\ u(x,t) = x(1 - e^{-0.06t}) + (x - 25e^{-0.06} - 1)e^{-0.06t}, \tag{4.10}$$

satisfies equation

$$\frac{\partial^\mu u}{\partial t^\mu} + 0.08(2 + \sin x)^2 x^2 \frac{\partial^2 u}{\partial x^2} + 0.06x\frac{\partial u}{\partial x} - 0.06u = 0, \ 0 < \mu \leq 1. \tag{4.11}$$

Proof. We have

$$\frac{\partial u}{\partial t} = (1.5)e^{-0.06-0.06t}, \ \ \frac{\partial u}{\partial x} = 1, \ \ \frac{\partial^2 u}{\partial x^2} = 0. \tag{4.12}$$

Substituting Eqs.(4.10) and (4.12) into the Eq.(4.11) gives (when $\mu = 1$)

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + 0.08(2+\sin x)^2 x^2 \frac{\partial^2 u}{\partial x^2} + 0.06x\frac{\partial u}{\partial x} - 0.06u \\
&= (1.5)e^{-0.06-0.06t} + [0.08(2+\sin x)^2 x^2](0) + 0.06x(1) \\
&\quad -0.06\left[x(1-e^{-0.06t}) + (x - 25e^{-0.06} - 1)e^{-0.06t}\right] \\
&= (1.5)e^{-0.06-0.06t} + 0 + 0.06x - 0.06x + 0.06xe^{-0.06t} \\
&\quad -0.06xe^{-0.06t} - 1.5e^{-0.06-0.06t} \\
&= 0.
\end{aligned}
$$

This completes the proof. □

5. Applications

Example 5.1. Consider the following fiactional Black-Scholes option pricing equation as

$$\frac{\partial^\mu u}{\partial t^\mu} = \frac{\partial^2 u}{\partial x^2} + (k-1)\frac{\partial u}{\partial x} - ku, \quad 0 < \mu \leq 1, \tag{5.1}$$

subject to the condition

$$u(x,0) = \max\left(e^x - 1, 0\right). \tag{5.2}$$

Applying Elzaki transform on both sides of equation (5.1) subject to the initial condition (5.2), we get

$$E[u(x,t)] = \max\left(e^x - 1, 0\right) + v^\mu E\left[u_{xx} + (k-1)u_x - ku\right]. \tag{5.3}$$

Using the inverse Elzaki transform on both sides of the equation (5.3) we have

$$u(x,t) = \max\left(e^x - 1, 0\right) - E^{-1}\left[v^\mu E\left[u_{xx} + (k-1)u_x - ku\right]\right].$$

Now appling HPM

$$u(x,t) = \max\left(e^x - 1, 0\right) - p\left[E^{-1}\left[v^\mu E\left[u_{xx} + (k-1)u_x - ku\right]\right]\right], \tag{5.4}$$

where $\max\left(e^x - 1, 0\right)$ represent the term arising from the source term and the prescribed initial condition. Now we apply the HPM

$$\sum_{n=0}^{\infty} p_n u_n(x,t) = \max(e^x - 1, 0) - p\left\{E^{-1}\left[v^\mu E\left(\sum_{n=0}^{\infty} p^n \mathrm{H}_n(u)\right)\right]\right\}, \tag{5.5}$$

where

$$H_n = u_{nxx} + (k-1)u_{nx} - ku_n, n \in N$$

and

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t).$$

Equating the corresponding power of p on both sides in equation (5.5), we have

$$
\begin{aligned}
p^o : u_0(x,t) &= \max\left(e^x - 1, 0\right), && (5.6)\\
p^1 : u_1(x,t) &= -E^{-1}\left\{v^\mu E\left[H_0(u)\right]\right\} && (5.7)\\
&= -\max(e^x,0)\frac{(-kt^\mu)}{\Gamma(\mu+1)} + \max\left(e^x-1,0\right)\frac{(-kt^\mu)}{\Gamma(\mu+1)},\\
p^2 : u_2(x,t) &= E^{-1}\left\{v^\mu E\left[H_1(u)\right]\right\} && (5.8)\\
&= -\max(e^x,0)\frac{(-kt^\mu)^2}{\Gamma(2\mu+1)} + \max\left(e^x-1,0\right)\frac{(-kt^\mu)^2}{\Gamma(2\mu+1)},\\
&\vdots\\
p^n : u_n(x,t) &= E^{-1}\left\{v^\mu E\left[H_{n-1}(u)\right]\right\} && (5.9)\\
&= -\max(e^x,0)\frac{(-kt^\mu)^n}{\Gamma(n\mu+1)} + \max\left(e^x-1,0\right)\frac{(-kt^\mu)^n}{\Gamma(n\mu+1)}.
\end{aligned}
$$

The solution $u(x,t)$ of Eq. (5.1) by using Eqs. (5.6)-(5.9) is given by

$$u(x,t) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n u_n(x,t),$$
$$u(x,t) = u_o(x,t) + u_1(x,t) + u_2(x,t) + \ldots,$$

$$
\begin{aligned}
u(x,t) = {}& \max\left(e^x,0\right)\\
&- \max(e^x,0)\left[1 + \frac{(-kt^\mu)}{\Gamma(\mu+1)} + \frac{(-kt^\mu)^2}{\Gamma(2\mu+1)} + \frac{(-kt^\mu)^3}{\Gamma(3\mu+1)} + \cdots\right]\\
&+ \max(e^x-1,0)\left[1 + \frac{(-kt^\mu)}{\Gamma(\mu+1)} + \frac{(-kt^\mu)^2}{\Gamma(2\mu+1)} + \frac{(-kt^\mu)^3}{\Gamma(3\mu+1)} + \cdots\right]\\
= {}& \max\left(e^x,0\right) - \max(e^x,0)\sum_{n=0}^{\infty}\frac{(-kt^\mu)^n}{\Gamma(n\mu+1)}\\
&+ \max(e^x-1,0)\sum_{n=0}^{\infty}\frac{(-kt^\mu)^n}{\Gamma(n\mu+1)},
\end{aligned}
$$

and the closed form

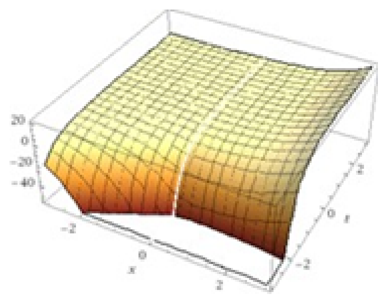
$$u(x,t) = \max\left(e^x,0\right)\left[1 - E_\mu\left(-kt^\mu\right)\right] + \max\left(e^x-1,0\right)\left[E_\mu\left(-kt^\mu\right)\right],$$

where $E_\mu(z)$ is Mittag-Leffler function in one parameter. When $\mu = 1$, then

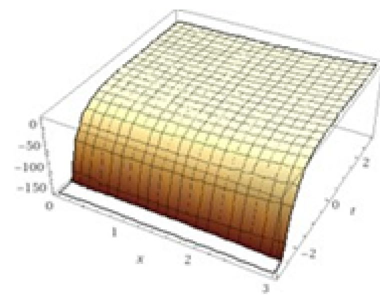
$$u(x,t) = \max\left(e^x,0\right)\left(1 - e^{-kt}\right) + \max\left(e^x-1,0\right)e^{-kt}. \tag{5.10}$$

Therefore, the obtained solution is (5.10).

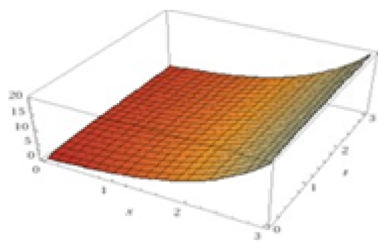
The following plots are obtained using wolfram Alpha software:



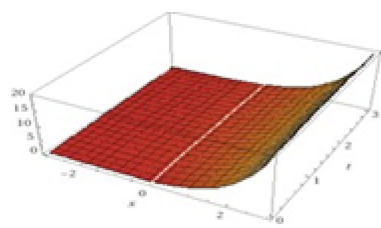
(A): $x = -3 \ldots 3, t = -3 \ldots 3, k = 2$



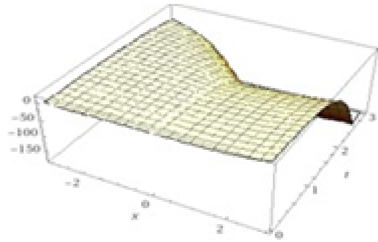
(B): $x = 0 \ldots 3, t = -3 \ldots 3, k = 2$



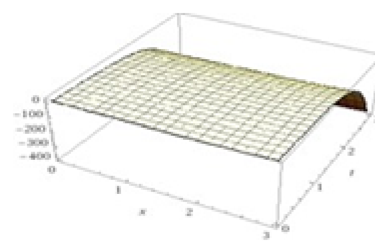
(C): $x = 0 \ldots 3, t = 0 \ldots 3, k = 2$



(D): $x = -3 \ldots 3, t = 0 \ldots 3, k = 2$



(E): $x = -3 \ldots 3, t = 0 \ldots 3; k = -2$



(F): $x = 0 \ldots 3, t = 0 \ldots 3, k = -2$

Example 5.2. Consider the another following generalized fractional Black Scholes equation as

$$\frac{\partial^\mu u}{\partial t^\mu} + 0.08(2 + \sin x)^2 x^2 \frac{\partial^2 u}{\partial x^2} + 0.06x\frac{\partial u}{\partial x} - 0.06u = 0 \ , \ 0 < \mu \leq 1, \tag{5.11}$$

subject to the initial condition

$$u(x, 0) = \max\left(x - 25e^{-0.06}, 0\right). \tag{5.12}$$

Applying Elzakit transform on both sides of equation (5.11) we have

$$E[u(x,t)] = \max\left(x-25e^{-0.06},0\right) - v^{\mu}E\left[0.08(2+\sin x)^2x^2u_{xx}+0.06xu_x-0.06u\right]. \quad (5.13)$$

Using the inverse Elzaki transform on both sides of the equation (5.13), we have

$$u(x,t) = \max\left(x-25e^{-0.06},0\right) - E^{-1}\left\{v^{\mu}E\left[0.08(2+\sin x)^2x^2u_{xx}+0.06xu_x-0.06u\right]\right\}.$$

Now by using HPM, we have

$$\sum_{n=0}^{\infty} p_n u_n(x,t) = \max(x-25e^{-0.06}-1,0) - p\left\{E^{-1}\left[v^{\mu}E\left(\sum_{n=0}^{\infty}p^n \mathrm{H}_n(u)\right)\right]\right\}, \quad (5.14)$$

where $H_n = 0.08\left(2+\sin x\right)^2 x^2 u_{nxx} + 0.06xu_{nx} - 0.06u_n, n \in N$. Equating the corresponding power of p on both sides in equation (5.14), we have

$$p^0 : u_0(x,t) = \max\left(x-25e^{-0.06},0\right), \quad (5.15)$$

$$p^1 : u_1(x,t) = E^{-1}\left\{v^{\mu}E\left[H_0(u)\right]\right\} = -x\frac{(-0.06t^{\mu})}{\Gamma(\mu+1)} + \max\left(x-25e^{-0.06},0\right)\frac{(-0.06t^{\mu})}{\Gamma(\mu+1)}, \quad (5.16)$$

$$p^2 : u_2(x,t) = E^{-1}\left\{v^{\mu}E\left[H_1(u)\right]\right\} = -x\frac{(-0.06t^{\mu})^2}{\Gamma(2\mu+1)} + \max\left(x-25e^{-0.06},0\right)\frac{(-0.06t^{\mu})^2}{\Gamma(2\mu+1)}, \quad (5.17)$$

$$\vdots$$

$$p^n : u_n(x,t) = E^{-1}\left\{v^{\mu}E\left[H_{n-1}(u)\right]\right\} = -x\frac{(-0.06t^{\mu})^n}{\Gamma(n\mu+1)} + \max\left(x-25e^{-0.06},0\right)\frac{(-0.06t^{\mu})^n}{\Gamma(n\mu+1)}. \quad (5.18)$$

The solution $u(x,t)$ of Eq. (5.11) by using Eqs. (5.15)-(5.18) is given by

$$u(x,t) = u_o(x,t) + u_1(x,t) + u_2(x,t) + \cdots,$$

$$u(x,t) = -x\left[\frac{(-0.06t^{\mu})}{\Gamma(\mu+1)} + \frac{(-0.06t^{\mu})^2}{\Gamma(2\mu+1)} + \frac{(-0.06t^{\mu})^3}{\Gamma(3\mu+1)} + \cdots\right] + \max(x-25e^{x},0)\left[1+\frac{(-0.06t^{\mu})}{\Gamma(\mu+1)}+\frac{(-0.06t^{\mu})^2}{\Gamma(2\mu+1)}+\frac{(-0.06t^{\mu})^3}{\Gamma(3\mu+1)}+\cdots\right],$$

$$u(x,t) = x\left[1-\sum_{n=0}^{\infty}\frac{(-0.06t^{\mu})^n}{\Gamma(n\mu+1)}\right]+\max\left(x-25e^{-0.06}-1,0\right)\sum_{n=0}^{\infty}\frac{(-0.06t^{\mu})^n}{\Gamma(n\mu+1)}$$

and closed form

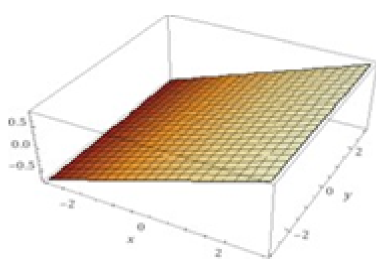
$$u(x,t) = x\left[1-E_{\mu}\left(-0.06t^{\mu}\right)\right]+\max\left(x-25e^{-0.06},0\right)\left[E_{\mu}\left(-0.06t^{\mu}\right)\right],$$

where $\mathrm{E}_{\mu}(\mathrm{z})$ is Mittag-Leffler function in one parameter. When $\mu = 1$, then

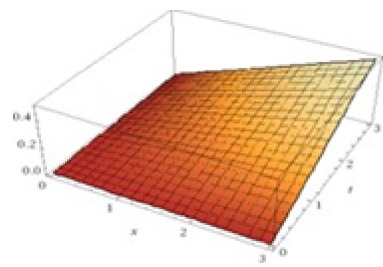
$$u(x,t) = x\left(1-e^{-0.06t}-1,0\right)+\max\left(x-25e^{-0.06}-1,0\right)e^{-0.06t}. \quad (5.19)$$

Therefore, the obtained solution is (5.19).

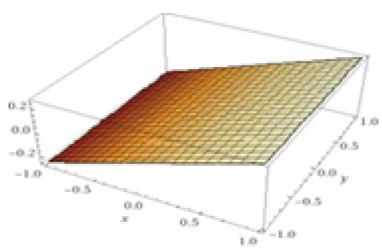
The following plots are obtained using wolform Alpha software:



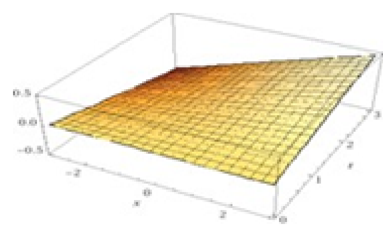
(G): $x = -3\ldots 3, t = 5$



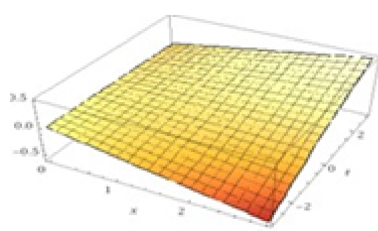
(H): $x = 0\ldots 3, t = 0\ldots 3$



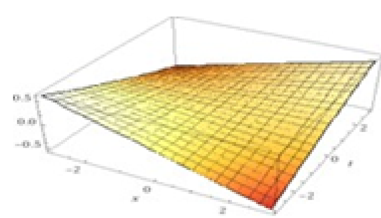
(I): $x = -1\ldots 1, t = 5$



(J): $x = -3\ldots 3, t = 0\ldots 3$



(K): $x = 0\ldots 3; t = -3\ldots 3$



(L): $x = -3\ldots 3, t = -3\ldots 3$

6. Conclusion

This paper intends to show the applicability of the Elzaki transform homotopy perturbation method to obtain an analytical solution for fractional Black-Scholes equation. It concludes that the new integral Elzaki transform homotopy perturbation method is very powerful, effective and efficient tool.

The obtained result, by this method are found to be more precise with the exact solution obtains by other existing methods. So the method could be one of the most practicable existing methods.

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