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ON INEQUALITIES CONCERNING COMPOSITE POLYNOMIALS

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Abstract. In this paper we consider a more general class of polynomials $P(R(z))$ of degree mr, where $R(z)$ is a polynomial of degree atmost r and prove compact generalizations of some well-know polynomial inequalities.

1. Introduction

Let P_n be the class of polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n and $P'(z)$ be its derivative, then

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The result is sharp and equality holds for the polynomials having all zeros at origin.

Inequality (1.1) is a famous result due to Bernstein [1], who proved it in 1912. Later, in 1930 he proved the following result from which inequality (1.1) can also be deduced.

Theorem 1.1. *Let $P(z)$ and $Q(z)$ be two polynomials with degree of $P(z)$ not exceeding that of $Q(z)$. If $Q(z)$ has all its zeros in $|z| \le 1$ and*

$$|P(z)| \le |Q(z)|, \quad \textit{for} \ \ |z| = 1,$$

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then

$$|P'(z)| \leq |Q'(z)|, \quad for \ \ |z| = 1. \tag{1.2}$$

Malik and Vong [4] improved Theorem 1.1 and replaced inequality (1.2) by

$$\left|\frac{zP'(z)}{n} + \beta\frac{P(z)}{2}\right| \leq \left|\frac{zQ'(z)}{n} + \beta\frac{Q(z)}{2}\right|, \tag{1.3}$$

for every β satisfying $|\beta| \leq 1$, n being the degree of $Q(z)$.

If we restrict ourselves to a class of polynomials having no zero in $|z| < 1$, then inequality (1.1), can be sharpened and we have for such class of polynomials

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2} \max_{|z|=1}|P(z)|. \tag{1.4}$$

Inequalitie (1.4) is sharp and equality holds for the polynomails having all their zeros on $|z| = 1$. Inequality (1.4) was conjectured by Erdös and later verified by Lax [3].

If $P(z)$ is a self-inverse polynomial, that is, if $P(z) = uQ(z),\ |u| = 1$, where $Q(z) = z^n\overline{(P\frac{1}{\bar{z}})}$, then it was proven by O'Hara and Rodrigues [5] that

$$\max_{|z|=1}|P'(z)| = \frac{n}{2} \max_{|z|=1}|P(z)|. \tag{1.5}$$

In this paper we consider the more generalized class of polynomials $P(R(z))$, introduced by Shah and Liman [6], where $R(z)$ is a polynomial of degree at most r defined by $(PoR)(z) = P(R(z))$, so that $PoR \in P_{nr}$ and prove the following results, which in turn generalize the above inequalities.

First we prove the following result which includes inequality (1.2) as a special case.

Theorem 1.2. *Let $PoR \in P_{nr}$ and $QoS \in P_{ms}$ be two composite polynomials with degree of P(R(z)) not exceeding that of Q(S(z)). If $Q(S(z)) \neq 0$ for $|z| > 1$, and*

$$|P(R(z))| \leq |Q(S(z))|, \quad for \ \ |z| = 1,$$

then

$$|P'(R(z))| \leq \frac{sM'}{rm'}|z|^{s-r}|Q'(S(z)|, \quad for \ \ |z| \geq 1, \tag{1.6}$$

where $m' = Min_{|z|=1}|R(z)|$ and $M' = Max_{|z|=1}|S(z)|$.

If we choose $R(z) = S(z)$ in inequality (1.6), we get the follwoing:

Corollary 1.3. *Let $PoR \in P_{nr}$ and $QoR \in P_{mr}$, such that $|P(R(z))| \leq |Q(R(z))|$ for $|z| = 1$. If $|Q(R(z))| \neq 0$ for $|z| > 1$, then*

$$|P'(R(z))| \leq \frac{M'}{m'}|Q'(R(z)|, \quad \textit{for} \ \ |z| \geq 1. \tag{1.7}$$

Remark 1.4. If in inequality (1.7) we take $R(z) = z$, so that $m' = M' = 1$, we get inequality (1.2).

Next we prove the following result which is of course improvement to the inequality (1.6).

Theorem 1.5. *Let $Q(S(z))$ be a polynomial of degree ns having all its zeros in $|z| \leq 1$ and $P(R(z))$ be a polynomial of degree not exceeding that of $Q(S(z))$. If $|P(R(z))| \leq |Q(S(z))|$ for $|z| = 1$, then for any $|\beta| < 1$,*

$$\left|\frac{zP'(R(z))R'(z)}{ns} + \frac{\beta}{2}P(R(z))\right| \leq \left|\frac{zQ'(S(z))S'(z)}{ns} + \frac{\beta}{2}Q(S(z))\right|. \tag{1.8}$$

For an approprate choice of argument of β in inequality (1.8), and making $|\beta| \to 1$, we get the following:

Corollary 1.6. *Let $Q(S(z))$ be a polynomials of degree ns having all its zeros in $|z| \leq 1$ and $P(R(z))$ be a polynomial of degree not exceeding that of $Q(S(z))$. If $|P(R(z))| \leq |Q(S(z))|$ for $|z| = 1$, then*

$$\left|\frac{P'(R(z))R'(z)}{ns}\right| + \left|\frac{Q(S(z))}{2}\right| \leq \left|\frac{Q'(S(z))S'(z)}{ns}\right| + \left|\frac{P(R(z))}{2}\right|. \tag{1.9}$$

If we choose $R(z) = S(z)$ in inequality (1.9), we get the following corollary:

Corollary 1.7. *Let $Q(R(z))$ be a polynomials of degree ns having all its zeros in $|z| \leq 1$ and $P(R(z))$ be a polynomial of degree not exceeding that of $Q(R(z))$. If $|P(R(z))| \leq |Q(R(z))|$ for $|z| = 1$, then*

$$\left|\frac{P'(R(z))R'(z)}{ns}\right| + \left|\frac{Q(R(z))}{2}\right| \leq \left|\frac{Q'(R(z))R'(z)}{ns}\right| + \left|\frac{P(R(z))}{2}\right|. \tag{1.10}$$

If we take $R(z) = z$ in inequality (1.10), we immediately have under the hypothesis of Theorem 1.1,

$$\left|\frac{P'(z)}{n}\right| + \left|\frac{Q(z)}{2}\right| \leq \left|\frac{Q'(z)}{n}\right| + \left|\frac{P(z)}{2}\right|, \quad \text{for} \ \ |z| = 1. \tag{1.11}$$

Inequality (1.11), is of course better than inequality (1.2) and has also been independently proved by Jain [2].

The following result that we prove will include inequality (1.4), as a particular case.

Theorem 1.8. *If $PoR \in P_{nr}$ and $P(R(z)) \neq 0$ for $|z| < 1$ and $R(z) \neq 0$ for $|z| \geq 1$, then for $|z| \geq 1$, we have*

$$|P'(R(z))| \leq \frac{M'n}{m'(m'+M')}|z|^{nr-r}|P(R(z))|, \tag{1.12}$$

where $m' = Min_{|z|=1}|R(z)|$ and $M' = Max_{|z|=1}|R(z)|$.

Remark 1.9. If we choose $R(z) = z$ in inequality (1.12), we get

$$|P'(z)| \leq \frac{n}{2}|z|^{n-1}Max_{|z|=1}|P(z))|, \quad \text{for} \ |z| \geq 1. \tag{1.13}$$

Which in particualr gives Erdös-Lax Theroem.

2. Lemmas

For the proof of above theorems we need the following lemma.

Lemma 2.1. *If $P(R(z))$ is a polynomial of degree nr having all its zeros in $|z| \leq 1$, then for $|z| = 1$,*

$$|z[P(R(z))]'| \geq \frac{ns}{2}|P(R(z))|.$$

Proof. Let z_i $(i = 1, 2, ..., ns)$ be the zeros of $P(R(z))$, then it is obvious

$$\left|e^{i\theta}\frac{[P(R(z))]'}{P(R(z))}\right| = \left|\sum_{i=1}^{ns}\frac{e^{i\theta}}{e^{i\theta}-z_i}\right| \geq \sum_{i=1}^{ns}\frac{1}{2} = \frac{ns}{2}. \tag{2.1}$$

Which concludes the proof of Lemma 2.1. □

3. Proof of theorems

Proof of Theorem 1.2. Since $Q(S(z)) \neq 0$ for $|z| > 1$, is a polynomial of degree ms and $|P(R(z))| \leq |Q(S(z))|$, for $|z| = 1$ where $|P(R(z))|$ is a polynomial of degree nr. Therefore, if β is any complex number with $|\beta| > 1$, then by Rouche's theorem all the zeros of $P(R(z)) - \beta Q(S(z))$ lie in $|z| \leq 1$. Hence, by Gauss-Lucas theorem all the zeros of $P'(R(z))R'(z) - \beta Q'(S(z))S'(z)$ lie in $|z| \leq 1$, for every complex number β with $|\beta| > 1$. This gives

$$|P'(R(z))||R'(z)| \leq |Q'(S(z)||S'(z)|, \quad \text{for} \ |z| \geq 1. \tag{3.1}$$

For if this is not true, then there is a point z_o with $|z_o| \geq 1$, such that

$$|P'(R(z_o))||R'(z_o)| > |Q'(S(z_o)||S'(z_o)|,$$

we take

$$\beta = \frac{P'(R(z_o))R'(z_o)}{Q'(S(z_o)S'(z_o)},$$

then $|\beta| > 1$ and with this choice of β, we have

$$P'(R(z_o))R'(z_o) - \beta Q'(S(z_o))S'(z_o) = 0, \text{ for } |z_o| \geq 1.$$

This is a contraduction and therefore

$$|P'(R(z))||R'(z)| \leq |Q'(S(z))||S'(z)|.$$

Let $R(z) \neq 0$ for $|z| \geq 1$. If $m' = Min_{|z|=1}|R(z)|$, then we can easly prove

$$|R'(z)| \geq rm'|z|^{r-1}, \quad \text{for } \ |z| \geq 1. \tag{3.2}$$

Similarly if $S(z) \neq 0$, for $|z| \geq 1$ and $Max_{|z|=1}|S(z)| = M'$, then

$$|S'(z)| \leq sM'|z|^{s-1}, \quad \text{for } \ |z| \geq 1. \tag{3.3}$$

Using inequalities (3.2) and (3.3) in inequality (3.1), we have

$$|P'(R(z))| \leq \frac{sM'}{rm'}|z|^{s-r}|Q'(S(z)|.$$

Which proves the result. □

Proof of Theorem 1.5**.** Let $P(R(z))$ and $Q(S(z))$ satisifies the hypothesis of the theorem. Therefore for any complex number α with $|\alpha| > 1$, we have by Rouche's Theorem all the zeros of $P(R(z)) + \alpha Q(S(z))$ lie in $|z| < 1$. Now by lemma 1 for $|z| = 1$, we have

$$\left|zP'(R(z))R'(z) + z\alpha Q'(S(z))S'(z)\right| \geq \frac{ns}{2}\left|P(R(z)) + \alpha Q(S(z))\right|. \tag{3.4}$$

From inequality (3.4), we note for any β with $|\beta| < 1$,

$$zP'(R(z))R'(z) + z\alpha Q'(S(z))S'(z)| + \beta\frac{ns}{2}(P(R(z)) + \alpha Q(S(z)) \neq 0. \tag{3.5}$$

From inequality (3.5), we conclude that

$$\frac{zP'(R(z))R'(z)}{ns} + \frac{\beta}{2}P(R(z)) \neq -\alpha\bigg(\frac{zQ'(S(z))S'(z)}{ns} + \frac{\beta}{2}Q(S(z))\bigg). \tag{3.6}$$

For an appropate choice of the argument of α in the right hand side of the inequality (3.6), we get

$$\left|\frac{zP'(R(z))R'(z)}{ns} + \frac{\beta}{2}P(R(z))\right| \neq |\alpha|\left|\frac{zQ'(S(z))S'(z)}{ns} + \frac{\beta}{2}Q(S(z))\right|. \tag{3.7}$$

From Inequality (3.7), we observe that

$$\left|\frac{zP'(R(z))R'(z)}{ns} + \frac{\beta}{2}P(R(z))\right| < |\alpha|\left|\frac{zQ'(S(z))S'(z)}{ns} + \frac{\beta}{2}Q(S(z))\right|. \tag{3.8}$$

Making $|\alpha| \to 1$, inequality (3.8) implies

$$\left|\frac{zP'(R(z))R'(z)}{ns} + \frac{\beta}{2}P(R(z))\right| \le \left|\frac{zQ'(S(z))S'(z)}{ns} + \frac{\beta}{2}Q(S(z))\right|.$$

Which completes the proof of Theorem 1.5. □

Proof of Theorem 1.8. Let $p(z) = P(R(z))$ and $q(z) = Q(R(z))$ such that $q(z) = z^{nr}\overline{p\left(\frac{1}{\bar{z}}\right)}$.

Now, we know

$$|p'(z)| + |q'(z)| \le nr|z|^{nr-1}Max_{|z|=1}|p(z)|, \quad \text{for} \quad |z| \ge 1.$$

Equivalently

$$|P'(R(z))||R'(z)| + |Q'(R(z))||R'(z)| \le nr|z|^{nr-1}Max_{|z|=1}|P(R(z))|. \quad (3.9)$$

Inequality (3.9), implies

$$\begin{aligned} &|P'(R(z))| + |Q'(R(z))| \\ &\le \frac{nr}{|R'(z)|}|z|^{nr-1}Max_{|z|=1}|P(R(z))|, \quad \text{for} \quad |z| \ge 1. \end{aligned} \quad (3.10)$$

Now, from inequality (1.7),

$$|P'(R(z))| + \frac{M'}{m'}|P'(R(z))| \le \frac{M'}{m'}(|P'(R(z))| + |Q'(R(z))|). \quad (3.11)$$

Using inequality (3.10) in inequality (3.11), we get

$$|P'(R(z))| \le \frac{M'nr}{|R'(z)|(m'+M')}|z|^{nr-1}|P(R(z))|, \quad \text{for} \quad |z| \ge 1.$$

$$|P'(R(z))| \le \frac{M'n}{m'(m'+M')}|z|^{(n-1)r}|P(R(z))|, \quad \text{for} \quad |z| \ge 1.$$

This completes the proof of Theorem 1.8. □

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