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SOLVABILITY OF A SYSTEM OF GENERALIZED NONLINEAR MIXED VARIATIONAL-LIKE INEQUALITIES

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Abstract. In this paper, we introduce and study a new system of generalized nonlinear mixed variational-like inequalities. By applying the Lemma of Ky Fan, we prove an existence theorem of solution of auxiliary problem for the system of generalized nonlinear mixed variational-like inequalities. By virtue of this existence result, we suggest and analyze an iterative method to compute the approximate solutions of the system of generalized nonlinear mixed variational-like inequalities and establish the convergence criteria of the iterative method. The results presented in this paper improve, extend and unify many known results in this area.

1. Introduction

Variational inequality theory has appeared as an effective and powerful tool to study and investigate a wide class of problems arising in pure and applied sciences including elasticity, optimization, economics, transportation

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and structural analysis see, e.g., [1, 2, 4–6, 9–15, 17–24] and the references therein. It is worth mentioning that the projection method and its variant forms cannot be extended for constructing iterative algorithms for variational-like inequalities, since it is not possible to find the projection. To overcome this drawback, one uses usually the auxiliary principle technique which does not depend on the projection mapping. This technique deals with finding a suitable auxiliary problem for the original problem. Further, this auxiliary problem is used to construct an algorithm for solving the original problem. Glowinski *et al.* [9] introduced this technique and used it to study the existence of a solution of mixed variational inequality. Later, Huang and Deng [10] and Zeng *et al.* [24] extended this technique to suggest and analyze a number of algorithms for solving various classes of variational inequalities.

In 1985, Pang [17] decomposed the original variational inequality problem into a system of variational inequality problems and discussed the convergence for system of variational inequality problems. Later, it was noticed that variational inequality problem over product of sets and the system of variational inequality problems both have same solution set, see for applications [3, 8]. Since then, many authors, see for example [1, 4, 8] studied the existence theory of various classes of system of variational inequality problems by exploiting fixed point theorems and minimax theorems. On the other hand, only a few iterative algorithms has been constructed for approximating the solution of system of variational inequality problems. Recently, Verma [19] studied the approximate solvability for a system of variational inequality problems based on system of projection methods.

Motivated and inspired by the research work going on in this field, we shall introduce and study consider a system of generalized nonlinear mixed variational-like inequalities problems and its related auxiliary problems in real Hilbert spaces. By the Lemma of Ky Fan [7], we prove an existence theorem of solution of auxiliary problem for the system of generalized nonlinear mixed variational-like inequalities. Further, by exploiting this theorem, we construct an algorithm for the system of generalized nonlinear mixed variational-like inequalities. Furthermore, we prove the existence of solution of the system of generalized nonlinear mixed variational-like inequalities and discuss the convergence analysis of the algorithm. The results presented in this paper improve, extend and unify many known results in this area.

2. Preliminaries

Throughout the paper unless otherwise stated, let $I = \{1, 2\}$ be an index set and for each $i \in I$, let H_i be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle_i$ and $\| \cdot \|_i$, respectively. For each $i \in I$, let K_i be a nonempty convex subset of H_i and $CB(H_i)$ be the family of all nonempty bounded closed subsets of H_i. For each $i \in I$, given single-valued mapping $N_i : H_1 \times H_2 \to H_i$, $\eta_i : K_i \times K_i \to H_i$, linear mapping $g_i : K_i \to K_i$, and set-valued mappings $A : K_1 \to CB(H_1)$, $T : K_2 \to CB(H_2)$. Now we consider the following system of generalized nonlinear mixed variational-like inequality problems: for given $(w_1^*, w_2^*) \in H_1 \times H_2$, find $(x, y) \in K_1 \times K_2$, $u \in Ax, v \in Ty$ such that

$$\begin{aligned} &\langle N_1(u,v) - w_1^*, \eta_1(g_1(s_1), g_1(x))\rangle_1 \\ &+ b_1(x, g_1(s_1)) - b_1(x, g_1(x)) \geq 0, \quad \forall\, s_1 \in K_1, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\langle N_2(u,v) - w_2^*, \eta_2(g_2(s_2), g_2(y))\rangle_2 \\ &+ b_2(y, g_2(s_2)) - b_2(y, g_2(y)) \geq 0, \quad \forall\, s_2 \in K_2, \end{aligned} \tag{2.2}$$

where for each $i \in I$, the bifunction $b_i : H_i \times H_i \to R$ satisfies the following properties:

(c1) b_i is linear in the first argument,

(c2) b_i is bounded, that is, there exists a constant $\gamma_i > 0$ such that

$$b_i(u_i, v_i) \leq \gamma_i \|u_i\|_i \|v_i\|_i, \quad \forall\, u_i, v_i \in H_i,$$

(c3) $b_i(u_i, v_i) - b_i(u_i, w_i) \leq b_i(u_i, v_i - w_i), \quad \forall\, u_i, v_i, w_i \in H_i$,

(c4) b_i is convex in the second argument.

Remark 2.1. ([10]) (1) For each $i \in I$, we have

$$|b_i(u_i, v_i)| \leq \gamma_i \|u_i\|_i \|v_i\|_i, \quad b_i(u_i, 0) = b_i(0, v_i) = 0, \quad \forall\, u_i, v_i \in H_i.$$

(2) For each $i \in I$, we have

$$|b_i(u_i, v_i) - b_i(u_i, w_i)| \leq \gamma_i \|u_i\|_i \|v_i - w_i\|_i, \quad \forall\, u_i, v_i, w_i \in H_i.$$

This implies that for each $i \in I$, b_i is continuous with respect to the second argument.

We need the following definitions, assumptions, lemma and known results in the sequel:

Definition 2.2. Let K be a nonempty convex subset of a real Hilbert space H. A set-valued mapping $A : K \to CB(H)$ is said to be *$\hat{H}$-Lipschitz continuous* if there exists a constant $\xi > 0$ such that

$$\hat{H}(A(x), A(y)) \leq \xi \|x - y\|, \quad \forall x, y \in H,$$

where $\hat{H}(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$.

Definition 2.3. Let $N : H \times H \to H$ be a nonlinear mapping and $A : K \to CB(H)$ be a set-valued mapping.

(1) N is said to be *Lipschitz continuous* in the first argument if there exists a constant $\alpha > 0$ such that

$$\|N(u,w) - N(v,w)\| \leq \alpha\|u - v\|, \quad \forall u, v, w \in H;$$

(2) N is said to be *strongly Lipschitz continuous* in the first argument with respect to A if there exists a constant $\beta > 0$ such that

$$\|x - y - (N(u,w) - N(v,w))\| \leq \beta\|x - y\|,$$

for all $w, x, y \in K, u \in Ax, v \in Ty$.

Similarly, we can define the Lipschitz continuity of N in the second argument.

Definition 2.4. Let $g : K \times K \to K$, a mapping $\eta : K \times K \to H$ is said to be

(1) *g-strongly monotone* if there exists a constant $\sigma > 0$ such that

$$\langle \eta(g(x), g(y)), x - y \rangle \geq \sigma\|x - y\|^2, \quad \forall x, y \in K,$$

(2) *Lipschitz continuous* if there exists a constant $\delta > 0$ such that

$$\|\eta(x,y)\| \leq \delta\|x - y\|, \quad \forall x, y \in K,$$

(3) g is said to be *Lipschitz continuous* if there exists a constant $a > 0$ such that

$$\|g(x) - g(y)\| \leq a\|x - y\|, \quad \forall x, y \in K.$$

Definition 2.5. Let D be a nonempty convex subset of a real Hilbert space H and $f : D \to (-\infty, +\infty]$ be a real functional.

(1) f is said to be *convex* if

$$f(\alpha u + (1 - \alpha)v) \leq \alpha f(u) + (1 - \alpha)f(v), \quad \forall u, v \in D, \ \alpha \in [0, 1],$$

(2) f is said to be *lower semicontinuous* on D if for each $\alpha \in (-\infty, +\infty]$, the set $\{u \in D : f(u) \leq \alpha\}$ is closed in D,

(3) f is said to be *concave* if $-f$ is convex,

(4) f is said to be *upper semicontinuous* on D if $-f$ is lower semicontinuous on D.

Lemma 2.6. ([7]) *Let B be a arbitrary nonempty subset in a topological vector space B and let $G : B \to 2^E$ be a KKM mapping. If $G(x)$ is closed for each $x \in B$ and is compact for at least one $x \in B$, then $\bigcap_{x \in B} G(x) \neq \emptyset$.*

Proposition 2.7. ([2]) *Let K be a nonempty convex subset of a real Hilbert space H and $f : K \to R$ be a lower semicontinuous and convex functional. Then f is weakly lower semicontinuous.*

Lemma 2.8. ([1, 2]) *Let X be a nonempty closed convex subset of a Hausdorff linear topological space E, and $\phi, \psi : X \times X \to R$ be mappings satisfying the following conditions:*

(a) $\psi(x, y) \leq \phi(x, y)$, $\forall x, y \in X$, *and* $\psi(x, x) \geq 0$, $\forall x \in X$;
(b) *for each* $x \in X$, $\phi(x, y)$ *is upper semicontinuous with respect to* y;
(c) *for each* $y \in X$, *the set* $\{x \in X : \psi(x, y) < 0\}$ *is a convex set;*
(d) *there exists a nonempty compact set* $K \subset X$ *and* $x_0 \in K$ *such that* $\psi(x_0, y) < 0$, $\forall y \in X \setminus K$;

Then there exists $\hat{y} \in K$ *such that* $\phi(x, \hat{y}) \geq 0$, $\forall x \in X$.

Assumption 2.9. *The mappings* $g : K \times K \to K$ *and* $\eta : K \times K \to H$ *satisfy the following conditions:*

(1) $\eta(x, y) = \eta(x, z) + \eta(z, y)$, $\forall x, y, z \in K$;
(2) $\eta(x, y)$ *is affine in the first argument,* $\forall x, y, z \in K$;
(3) *for an given* $u, , y$, $x \mapsto \langle N(u, v), \eta(y, g(x))\rangle$ *is continuous from the weak topology to the weak topology.*

3. Auxiliary problem and algorithm

For given $(w_1^*, w_2^*) \in H_1 \times H_2$ and $(x_1, x_2) \in K_1 \times K_2, u \in Ax_1, v \in Tx_2$, we consider the following problem $P_1(u, v, x_1, x_2)$: find $(z_1, z_2) \in K_1 \times K_2$ such that

$$\begin{aligned}\langle z_1, s_1 - z_1\rangle_1 \geq \langle x_1, s_1 - z_1\rangle_1 - \rho\langle N_1(u, v) - w_1^*, \eta_1(g_1(s_1), g_1(z_1))\rangle_1 \\ + \rho b_1(x_1, g_1(z_1)) - \rho b_1(x_1, g_1(s_1)), \quad \forall s_1 \in K_1,\end{aligned} \tag{3.1}$$

$$\begin{aligned}\langle z_2, s_2 - z_2\rangle_2 \geq \langle x_2, s_1 - z_2\rangle_1 - \rho\langle N_2(u, v) - w_2^*, \eta_2(g_2(s_2), g_2(z_2))\rangle_2 \\ + \rho b_2(x_2, g_2(z_2)) - \rho b_2(x_2, g_2(s_2)), \quad \forall s_2 \in K_2,\end{aligned} \tag{3.2}$$

where $\rho > 0$ is a constant.

Theorem 3.1. *For each* $i \in I$, *let* K_i *be a nonempty bounded closed subset of a real Hilbert space* H_i, *linear mapping* $g_i : K_i \to K_i$, *bifunction* $b_i(\cdot, \cdot)$ *satisfies the condtions* (c1)$\sim$(c4), *and Assumption* 2.9 *holds. Then the auxiliary problem* $P_1(u, v, x_1, x_2)$ *has a solution.*

Proof. For each $i \in I$, given $w_i^* \in H_i, x_i \in K_i, u \in Ax_1, v \in Tx_2$, we define the mapping $G_i : K_i \to 2^{H_i}$ by

$$\begin{aligned}G_i(s_i) = \big\{z_i \in K_i : \langle z_i - x_i, s_i - z_i\rangle_i + \rho[\langle N_i(u, v) - w_i^*, \eta_i(g_i(s_i), g_i(z_i))\rangle_i \\ + b_i(x_i, g_i(s_i)) - b_i(x_i, g_i(z_i))] \geq 0\big\}, \quad \forall s_i \in K_i.\end{aligned}$$

Note that for each $s_i \in K_i$, $G_i(s_i)$ is nonempty, since $s_i \in G_i(s_i)$.

We shall prove that G_i is a KKM mapping. Suppose that there is a finite subset $\{s_{i1}, s_{i2}, \ldots, s_{ik}\}$ of K_i and that $\alpha_{ij} \geq 0$ for $j \in \{1, 2, ..., k\}$ with $\sum_{j=1}^{k} \alpha_{ij} = 1$ such that $\hat{z}_i = \sum_{j=1}^{k} \alpha_{ij} s_{ij} \notin G_i(s_{ij})$ for all j. Then we have

$$\begin{aligned}&\langle \hat{z}_i - x_i, s_{ij} - \hat{z}_i \rangle_i + \rho[\langle N_i(u, v) - w_i^*, \eta_i(g_i(s_{ij}), g_i(\hat{z}_i)) \rangle_i \\ &+ b_i(x_i, g_i(s_{ij})) - b_i(x_i, g_i(\hat{z}_i))] < 0, \quad \forall j.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{j=1}^{k} \alpha_{ij} \langle \hat{z}_i - x_i, s_{ij} - \hat{z}_i \rangle_i + \rho \sum_{j=1}^{k} \alpha_{ij} \langle N_i(u, v) - w_i^*, \eta_i(g_i(s_{ij}), g_i(\hat{z}_i)) \rangle_i \\ &+ \rho \sum_{j=1}^{k} \alpha_{ij} [b_i(x_i, g_i(s_{ij})) - b_i(x_i, g_i(\hat{z}_i))] < 0.\end{aligned}$$

From Assumption 2.9(1), we have $\eta_i(x, x) = 0$, $\forall x \in K_i$. By using the convexity of $b_i(\cdot, \cdot)$ in the second argument, Assumption 2.9(2) and g is linear, we get

$$\begin{aligned}0 &= \langle \hat{z}_i - x_i, \hat{z}_i - \hat{z}_i \rangle_i + \rho \langle N_i(u, v) - w_i^*, \eta_i(g_i(\hat{z}_i), g_i(\hat{z}_i)) \rangle_i \\ &\quad + \rho[b_i(x_i, g_i(\hat{z}_i)) - b_i(x_i, g_i(\hat{z}_i))] < 0,\end{aligned}$$

which is a contradiction. Hence, G_i is a KKM mapping.

Since $\overline{G_i(s_i)}^w$ [the weak closure of $G_i(s_i)$] is a weakly closed subset of a bounded set K_i in H_i, it is weakly compact. Hence, by Lemma 2.6, we have $\bigcap_{s_i \in K_i} \overline{G_i(s_i)}^w \neq \emptyset$.

Let $z_i \in \bigcap_{s_i \in K_i} \overline{G_i(s_i)}^w$. Then for each $s_i \in K_i$, there exists a sequence $\{z_{im}\}$ in $G_i(s_i)$ such that $z_{im} \to z_i$ weakly. Hence we have

$$\begin{aligned}&\langle z_{im} - x_i, s_i - z_{im} \rangle_i + \rho[\langle N_i(u, v) - w_i^*, \eta_i(g_i(s_i), g_i(z_{im})) \rangle_i \\ &+ b_i(x_i, g_i(s_i)) - b_i(x_i, g_i(z_{im}))] \geq 0.\end{aligned} \tag{3.3}$$

Now, since the $\| \cdot \|_i$ is weakly lower semicontinuous, we have

$$\begin{aligned}&\limsup_{m \to \infty} \langle z_{im} - x_i, s_i - z_{im} \rangle_i \\ &= \limsup_{m \to \infty} [\langle z_{im} - x_i, s_i \rangle_i + \langle x_i, z_{im} \rangle_i + \|z_{im}\|_i] \\ &\leq \lim_{m \to \infty} \langle z_{im} - x_i, s_i \rangle_i + \lim_{m \to \infty} \langle x_i, z_{im} \rangle_i - \liminf_{m \to \infty} \|z_{im}\|_i \\ &\leq \langle z_i - x_i, s_i - z_i \rangle_i.\end{aligned}$$

Since $b_i(\cdot, \cdot)$ is convex and continuous in the second argument, it is weakly lower semicontinuous in the second argument. Thus, it follows from (3.3) and

Assumption 2.9(3) that

$$\begin{aligned}&\langle z_i - x_i, s_i - z_i\rangle_i + \rho[\langle N_i(u,v) - w_i^*, \eta_i(g_i(s_i), g_i(z_i))\rangle_i \\ &\quad + b_i(x_i, g_i(s_i)) - b_i(x_i, g_i(z_i))] \\ &\geq \limsup_{m\to\infty}\{\langle z_{im} - x_i, s_i - z_{im}\rangle_i + \rho[\langle N_i(u,v) - w_i^*, \eta_i(g_i(s_i), g_i(z_{im}))\rangle_i \\ &\quad + b_i(x_i, g_i(s_i)) - b_i(x_i, g_i(z_{im}))]\} \geq 0,\end{aligned}$$

and hence

$$\begin{aligned}\langle z_i, s_i - z_i\rangle_i \geq \langle x_i, s_i - z_i\rangle_i - \rho\langle N_i(u,v) - w_i^*, \eta_i(g_i(s_i), g_i(z_i))\rangle_i \\ + \rho b_i(x, g_i(z_i)) - \rho b_i(x, g_i(s_i)), \quad \forall\, s_i \in K_i.\end{aligned}$$

This shows that the auxiliary problem $P_1(u, v, x_1, x_2)$ has a solution. □

By using Theorem 3.1, we now construct the algorithm for solving the system of generalized nonlinear mixed variational-like inequalities (2.1) and (2.2).

Algorithm 3.2. *For given* $(w_1^*, w_2^*) \in H_1 \times H_2$ *and* $(x_0, y_0) \in K_1 \times K_2, u_0 \in Ax_0, v_0 \in Ty_0$, *there exist the sequence* $\{u_n\}_{n\geq 0} \subset H_1$, $\{v_n\}_{n\geq 0} \subset H_2$, *and* $\{(x_n, y_n)\}_{n\geq 0} \subset K_1 \times K_2$ *satisfying the following conditions:*

$$\begin{aligned}u_n \in Ax_n, \quad \|u_n - u_{n+1}\|_1 \leq \left(1 + \frac{1}{n+1}\right)\hat{H}(Ax_n, Ax_{n+1}), \\ v_n \in Ty_n, \quad \|v_n - v_{n+1}\|_2 \leq \left(1 + \frac{1}{n+1}\right)\hat{H}(Ty_n, Ty_{n+1}),\end{aligned}$$

and

$$\begin{aligned}&\langle x_{n+1}, s_1 - x_{n+1}\rangle_1 \\ &\geq \langle x_n, s_1 - x_{n+1}\rangle_1 - \rho\langle N_1(u_n, v_n) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1 \\ &\quad + \rho b_1(x_n, g_1(x_{n+1})) - \rho b_1(x_n, g_1(s_1)), \quad \forall\, s_1 \in K_1,\, n \geq 0,\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\langle y_{n+1}, s_2 - y_{n+1}\rangle_2 \\ &\geq \langle y_n, s_1 - y_{n+1}\rangle_1 - \rho\langle N_2(u_n, v_n) - w_2^*, \eta_2(g_2(s_2), g_2(y_{n+1}))\rangle_2 \\ &\quad + \rho b_2(y_n, g_2(y_{n+1})) - \rho b_2(y_n, g_2(s_2)), \quad \forall\, s_2 \in K_2,\, n \geq 0,\end{aligned} \tag{3.5}$$

where $\rho > 0$ *is a constant.*

In the next section, we extend the auxiliary principle technique of Glowinski *et al.* [1] to study the the system of generalized nonlinear mixed variational-like inequalities (2.1) and (2.2). We give an existence theorem of a solution of the auxiliary problem for the the system of generalized nonlinear mixed variational-like inequalities (2.1) and (2.2).

Based on this existence theorem, we construct an iterative algorithm for the the system of generalized nonlinear mixed variational-like inequalities (2.1) and (2.2).

For given $(w_1^*, w_2^*) \in H_1 \times H_2$ and $(x_1, x_2) \in K_1 \times K_2, u \in Ax_1, v \in Tx_2$, we consider the following problem $P_2(u, v, x_1, x_2)$: find $(z_1, z_2) \in K_1 \times K_2$ such that

$$\begin{aligned} &\langle g_1(z_1), s_1 - z_1\rangle_1 \\ &\geq \langle g_1(x_1), s_1 - z_1\rangle_1 - \rho\langle N_1(u, v) - w_1^*, \eta_1(g_1(s_1), g_1(z_1))\rangle_1 \\ &\quad + \rho b_1(x_1, g_1(z_1)) - \rho b_1(x_1, g_1(s_1)), \quad \forall s_1 \in K_1, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\langle g_2(z_2), s_2 - z_2\rangle_2 \\ &\geq \langle g_2(x_2), s_2 - z_2\rangle_2 - \rho\langle N_2(u, v) - w_2^*, \eta_2(g_2(s_2), g_2(z_2))\rangle_2 \\ &\quad + \rho b_2(x_2, g_2(z_2)) - \rho b_2(x_2, g_2(s_2)), \quad \forall s_2 \in K_2. \end{aligned} \tag{3.7}$$

Theorem 3.3. *For each $i \in I$, let $g_i : K_i \to K_i$, be Lipschitz continuous and strongly monotone with constants $a_i > 0$ and $b_i > 0$, respectively; $b_i(\cdot, \cdot)$ satisfies the conditions* (c1)$\sim$(c4), *$\eta_i : K_i \times K_i \to H_i$ satisfies Assumption* 2.9 *and Lipschitz continuous with constants $\delta_i > 0$. Then the auxiliary problem $P_2(u, v, x_1, x_2)$ has a solution.*

Proof. Define the functionals ϕ_i and $\psi_i : K_i \times K_i \to R$ by

$$\begin{aligned} \phi_i(s_i, z_i) = &\langle g_i(s_i), s_i - z_i\rangle_i - \langle g_i(x_i), s_i - z_i\rangle_i \\ &+ \rho\langle N_i(u, v) - w_i^*, \eta_i(g_i(s_i), g_i(z_i))\rangle_i \\ &- \rho b_i(x_i, g_i(z_i)) + \rho b_i(x_i, g_i(s_i)) \end{aligned}$$

and

$$\begin{aligned} \psi_i(s_i, z_i) = &\langle g_i(z_i), s_i - z_i\rangle_i - \langle g_i(x_i), s_i - z_i\rangle_i \\ &+ \rho\langle N_i(u, v) - w_i^*, \eta_i(g_i(s_i), g_i(z_i))\rangle_i \\ &- \rho b_i(x_i, g_i(z_i)) + \rho b_i(x_i, g_i(s_i)) \end{aligned}$$

for all $s_i, z_i \in K_i$, respectively. We shall prove that the mappings ϕ_i, ψ_i, satisfy all the conditions of Lemma 2.8 in the weak topology.

Indeed, dearly ϕ_i and ψ_i satisfy condition (a) of Lemma 2.8. From property (c4) of b, Remark 2.1(2) and the Lipschitz continuity of g, it follows that $b_i(x_i, g_i(z_i))$ is convex and Lipschitz continuous with respect to z_i. Again from Assumption 2.9(3), it follows that the function

$$z_i \longmapsto \langle N_i(u, v) - w_i^*, \eta_i(g_i(s_i), g_i(z_i))\rangle_i$$

is concave and upper semicontinuous. Therefore, we conclude that $\psi_i(s_i, z_i)$ is weakly upper semicontinuous with respect to z_i. Now we show that the set $\{s_i \in K_i : \phi_i(s_i, z_i) < 0\}$ is a convex set for each $z_i \in K_i$. Indeed, suppose that $\{s_{i1}, s_{i2}, \ldots, s_{ik}\}$ is a finite set of $\{s_i \in K_i : \phi_i(s_i, z_i) < 0\}$ and that $\alpha_{ij} \geq 0$ for

$j \in \{1, 2, ..., k\}$ with $\sum_{j=1}^{k} \alpha_{ij} = 1$. Then we write $\hat{s}_i = \sum_{j=1}^{k} \alpha_{ij} s_{ij}$. Observe that for all j,

$$\langle g_i(z_i) - g_i(x_i), s_i - z_i \rangle_i + \rho \langle N_i(u, v) - w_i^*, \eta_i(g_i(s_i), g_i(z_i)) \rangle_i \\ - \rho b_i(x_i, g_i(z_i)) + \rho b_i(x_i, g_i(s_i)) < 0$$

and hence

$$\begin{aligned} 0 > & \sum_{j=1}^{k} \alpha_{ij} \langle g_i(z_i) - g_i(x_i), s_i j - z_i \rangle_i \\ & + \rho \sum_{j=1}^{k} \alpha_{ij} \langle N_i(u, v) - w_i^*, \eta_i(g_i(s_i j), g_i(z_i)) \rangle_i \\ & - \rho b_i(x_i, g_i(z_i)) + \rho \sum_{j=1}^{k} \alpha_{ij} b_i(x_i, g_i(s_i j)) \\ \geq & \langle g_i(z_i) - g_i(x_i), \hat{s}_i - z_i \rangle_i + \rho \langle N_i(u, v) - w_i^*, \eta_i(g_i(\hat{s}_i), g_i(z_i)) \rangle_i \\ & - \rho b_i(x_i, g_i(z_i)) + \rho b_i(x_i, g_i(\hat{s}_i)). \end{aligned}$$

This implies that $\hat{s}_i \in \{s_i \in K_i : \phi_i(s_i, z_i) < 0\}$. Therefore, conditions (b) and (c) of Lemma 2.8 hold. Finally we shall prove that condition (d) of Lemma 2.8 holds. Indeed, let

$$\omega_i = b_i^{-1}[a_i \|_i \|x_i\|_i + \rho \delta_i a_i \|N_i(u, v) - w_i^*\|_i + \rho \gamma_i a_i \|x_i\|_i], \\ T_i = \{z_i \in K_i : \|z_i\|_i \leq \omega_i\}.$$

Then T_i is a weakly compact subset of K_i. For any fixed $z_i \in K_i \setminus T_i$, take $s_{i0} \in T_i$. From Assumption 2.9, the Lipschitz continuity of g_i, η_i and the strongly monotone of g_i, and Remark 2.6(2), we have

$$\begin{aligned} & \psi_i(s_{i0}, z_i) \\ & = \psi_i(0, z_i) \\ & = -\langle g_i(z_i), z_i \rangle_i + \langle g_i(x_i), z_i \rangle_i + \rho \langle N_i(u, v) - w_i^*, \eta_i(g_i(0), g_i(z_i)) \rangle_i \\ & \quad + \rho b_i(x_i, g_i(z_i)) + \rho b_i(x_i, g_i(0)) \\ & = \langle g_i(0) - g_i(z_i), z_i - 0 \rangle_i + \langle g_i(x_i) - g_i(0), z_i \rangle_i \\ & \quad + \rho \langle N_i(u, v) - w_i^*, \eta_i(g_i(0), g_i(z_i)) \rangle_i - \rho b_i(x_i, g_i(z_i)) + \rho b_i(x_i, g_i(0)) \\ & \leq -b_i \|z_i\|_i^2 + \|g_i(x_i) - g_i(0)\|_i \|z_i\|_i \\ & \quad + \rho \|N_i(u, v) - w_i^*\|_i \|\eta_i(g_i(0), g_i(z_i))\|_i + \rho \gamma_i a_i \|z_i\|_i \|x_i\|_i \\ & = -\|z_i\|_i \{b_i \|z_i\|_i - [a_i \|_i \|x_i\|_i + \rho \delta_i a_i \|N_i(u, v) - w_i^*\|_i + \rho \gamma_i a_i \|x_i\|_i]\}. \end{aligned}$$

Therefore Condition (d) of Lemma 2.8 holds. By Lemma 2.8, there exists a $\bar{z}_i \in K_i$ such that $\phi_i(s_i, \bar{z}_i) \geq 0$, $\forall s_i \in K_i$, that is,

$$\begin{aligned} &\langle g_i(s_i), s_i - \bar{z}_i\rangle_i - \langle g_i(x_i), s_i - \bar{z}_i\rangle_i + \rho\langle N_i(u,v) - w_i^*, \eta_i(g_i(s_i), g_i(\bar{z}_i))\rangle_i \\ &- \rho b_i(x_i, g_i(\bar{z}_i)) + \rho b_i(x_i, g_i(s_i)) \geq 0, \quad \forall s_i \in K_i. \end{aligned} \tag{3.8}$$

For arbitrary $t_i \in (0,1]$ and $s_i \in K_i$, let $x_{t_i} = t_i s_i + (1-t_i)\bar{z}_i$. Replacing s_i by $\bar{z}_i$ in (3.8) and utilizing Assumption 2.9(3) and Property (iv) of b_i, we obtain

$$\begin{aligned} 0 &\leq \langle g_i(x_{t_i}), x_{t_i} - \bar{z}_i\rangle_i - \langle g_i(x_i), x_{t_i} - \bar{z}_i\rangle_i \\ &\quad + \rho\langle N_i(u,v) - w_i^*, \eta_i(g_i(x_{t_i}), g_i(\bar{z}_i))\rangle_i - \rho b_i(x_i, g_i(\bar{z}_i)) + \rho b_i(x_i, g_i(x_{t_i})) \\ &= t_i\langle g_i(x_{t_i}), s_i - \bar{z}_i\rangle_i - t_i\langle g_i(x_i), s_i - \bar{z}_i\rangle_i \\ &\quad - \rho\langle N_i(u,v) - w_i^*, \eta_i(g_i(\bar{z}_i), g_i(t_i s_i + (1-t_i)\bar{z}_i))\rangle_i \\ &\quad - \rho b_i(x_i, g_i(\bar{z}_i)) + \rho b_i(x_i, g_i(t_i s_i + (1-t_i)\bar{z}_i)) \\ &\leq t_i\langle g_i(x_{t_i}), s_i - \bar{z}_i\rangle_i - t_i\langle g_i(x_i), s_i - \bar{z}_i\rangle_i \\ &\quad + \rho t_i\langle N_i(u,v) - w_i^*, \eta_i(g_i(s_i), g_i(\bar{z}_i))\rangle_i + \rho t_i[b_i(x_i, g_i(s_i)) - b_i(x_i, g_i(\bar{z}_i))]. \end{aligned}$$

Hence,

$$\begin{aligned} &\langle g_i(x_{t_i}), s_i - \bar{z}_i\rangle_i - \langle g_i(x_i), s_i - \bar{z}_i\rangle_i + \rho\langle N_i(u,v) - w_i^*, \eta_i(g_i(s_i), g_i(\bar{z}_i))\rangle_i \\ &+ \rho[b_i(x_i, g_i(s_i)) - b_i(x_i, g_i(\bar{z}_i))] \geq 0, \end{aligned}$$

and consequently,

$$\begin{aligned} \langle g_i(x_{t_i}), s_i - \bar{z}_i\rangle_i \geq{}& \langle g_i(x_i), s_i - \bar{z}_i\rangle_i - \rho\langle N_i(u,v) - w_i^*, \eta_i(g_i(s_i), g_i(\bar{z}_i))\rangle_i \\ &+ \rho b_i(x_i, g_i(\bar{z}_i)) - \rho b_i(x_i, g_i(s_i)). \end{aligned}$$

Letting $t_i \longrightarrow 0^+$, we have

$$\begin{aligned} \langle g_i(\bar{z}_i), s_i - \bar{z}_i\rangle_i \geq{}& \langle g_i(x_i), s_i - \bar{z}_i\rangle_i - \rho\langle N_i(u,v) - w_i^*, \eta_i(g_i(s_i), g_i(\bar{z}_i))\rangle_i \\ &+ \rho b_i(x_i, g_i(\bar{z}_i)) - \rho b_i(x_i, g_i(s_i)), \quad \forall s_i \in K_i. \end{aligned}$$

Therefore, $\bar{z}_i \in K_i$ is a solution of the auxiliary problem $P_2(u, v, x_1, x_2)$. This completes the proof. □

By using Theorem 3.3, we now construct the algorithm for solving the system of generalized nonlinear mixed variational-like inequalities (2.1) and (2.2).

Algorithm 3.4. *For given* $(w_1^*, w_2^*) \in H_1 \times H_2$ *and* $(x_0, y_0) \in K_1 \times K_2, u_0 \in Ax_0, v_0 \in Ty_0$, *there exist the sequence* $\{u_n\}_{n\geq 0} \subset H_1, \{v_n\}_{n\geq 0} \subset H_2$, *and*

$\{(x_n, y_n)\}_{n\geq 0} \subset K_1 \times K_2$ *satisfying the following conditions:*

$$u_n \in Ax_n, \quad \|u_n - u_{n+1}\|_1 \leq \left(1 + \frac{1}{n+1}\right)\hat{H}(Ax_n, Ax_{n+1}),$$
$$v_n \in Ty_n, \quad \|v_n - v_{n+1}\|_2 \leq \left(1 + \frac{1}{n+1}\right)\hat{H}(Ty_n, Ty_{n+1}),$$

and

$$\begin{aligned}&\langle g_1(x_{n+1}), s_1 - x_{n+1}\rangle_1 \\ &\geq \langle g_1(x_n), s_1 - x_{n+1}\rangle_1 - \rho\langle N_1(u_n, v_n) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1 \\ &\quad + \rho b_1(x_n, g_1(x_{n+1})) - \rho b_1(x_n, g_1(s_1)), \quad \forall\, s_1 \in K_1,\ n \geq 0,\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\langle g_2(y_{n+1}), s_2 - y_{n+1}\rangle_2 \\ &\geq \langle g_2(y_n), s_1 - y_{n+1}\rangle_1 - \rho\langle N_2(u_n, v_n) - w_2^*, \eta_2(g_2(s_2), g_2(y_{n+1}))\rangle_2 \\ &\quad + \rho b_2(y_n, g_2(y_{n+1})) - \rho b_2(y_n, g_2(s_2)), \quad \forall\, s_2 \in K_2,\ n \geq 0,\end{aligned} \tag{3.10}$$

where $\rho > 0$ is a constant.

4. Existence and convergence theorem

Theorem 4.1. *For each $i \in I$, let K_i be a nonempty convex subset of H_i and bifunction $b_i(\cdot,\cdot)$ satisfies the conditions* (c1)∼(c4). *Let $N_i : H_1 \times H_2 \to H_i$ be strongly Lipschitz continuous in the first argument and Lipschitz continuous in the second argument with constants $\alpha_i > 0$ and $\beta_i > 0$, respectively; set-valued mappings $A : K_1 \to CB(H_1)$, $T : K_2 \to CB(H_2)$ be $\hat{H}$-Lipschitz continuous with constants $\xi_1 > 0$ and $\xi_2 > 0$, respectively; linear mapping $g_i : K_i \to K_i$, be Lipschitz continuous with constants $a_i > 0$; and $\eta_i : K_i \times K_i \to H_i$ satisfies Assumption* 2.9 *and η_i be g_i-strongly monotone with constants $\sigma_i > 0$, Lipschitz continuous with constants $\delta_i > 0$. If there exists a constant $\rho > 0$ such that*

$$\begin{aligned}&0 < \rho < \min\left\{\frac{\sigma_i - (t_i\alpha_1 + c_i\epsilon_i)}{t_i^2 - (t_i\alpha_1 + c_i\epsilon_i)^2}, \frac{1}{t_i\alpha_1 + c_i\epsilon_i}\right\}, \\ &\quad t_i\alpha_1 + c_i\epsilon_i < \sigma_i < t_i,\end{aligned} \tag{4.1a}$$

where

$$t_i = \delta_i a_i, \quad c_i = \beta_i\xi_i, \quad \epsilon_1 = \frac{\gamma_1 a_1}{c_1} + t_2, \quad \epsilon_2 = \frac{\gamma_2 a_1}{c_2} + t_1, \quad i \in I,$$

then there are $(\hat{x}, \hat{y}) \in K_1 \times K_2$, $\hat{u} \in A\hat{x}$, $\hat{v} \in T\hat{y}$ satisfying the system of generalized nonlinear mixed variational-like inequalities (2.1) *and* (2.2), *and*

$$(x_n, y_n) \to (\hat{x}, \hat{y}), \quad u_n \to \hat{u}, \quad v_n \to \hat{v}, \quad n \to \infty,$$

where $\{(x_n, y_n)\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$ are defined by Algorithm 3.2.

Proof. Using Algorithm 3.2, we obtain that

$$\begin{aligned}&\langle x_n, s_1 - x_n\rangle_1\\ &\geq \langle x_{n-1}, s_1 - x_n\rangle_1 - \rho\langle N_1(u_{n-1}, v_{n-1}) - w_1^*, \eta_1(g_1(s_1), g_1(x_n))\rangle_1 \\ &\quad + \rho b_1(x_{n-1}, g_1(x_n)) - \rho b_1(x_{n-1}, g_1(s_1)),\end{aligned} \tag{4.2a}$$

$$\begin{aligned}&\langle y_n, s_2 - y_n\rangle_2\\ &\geq \langle y_{n-1}, s_1 - y_n\rangle_1 - \rho\langle N_2(u_{n-1}, v_{n-1}) - w_2^*, \eta_2(g_2(s_2), g_2(y_n))\rangle_2 \\ &\quad + \rho b_2(y_{n-1}, g_2(y_n)) - \rho b_2(y_{n-1}, g_2(s_2)),\end{aligned} \tag{4.3a}$$

$$\begin{aligned}&\langle x_{n+1}, s_1 - x_{n+1}\rangle_1\\ &\geq \langle x_n, s_1 - x_{n+1}\rangle_1 - \rho\langle N_1(u_n, v_n) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1 \\ &\quad + \rho b_1(x_n, g_1(x_{n+1})) - \rho b_1(x_n, g_1(s_1)),\end{aligned} \tag{4.4a}$$

$$\begin{aligned}&\langle y_{n+1}, s_2 - y_{n+1}\rangle_2\\ &\geq \langle y_n, s_1 - y_{n+1}\rangle_1 - \rho\langle N_2(u_n, v_n) - w_2^*, \eta_2(g_2(s_2), g_2(y_{n+1}))\rangle_2 \\ &\quad + \rho b_2(y_n, g_2(y_{n+1})) - \rho b_2(y_n, g_2(s_2))\end{aligned} \tag{4.5a}$$

for all $n \geq 1$. Taking $s_1 = x_{n+1}$ in (4.2a) and $s_1 = x_n$ in (4.4a), we conclude that

$$\begin{aligned}&\langle x_n, x_{n+1} - x_n\rangle_1\\ &\geq \langle x_{n-1}, x_{n+1} - x_n\rangle_1 - \rho\langle N_1(u_{n-1}, v_{n-1}) - w_1^*, \eta_1(g_1(x_{n+1}), g_1(x_n))\rangle_1 \\ &\quad + \rho b_1(x_{n-1}, g_1(x_n)) - \rho b_1(x_{n-1}, g_1(x_{n+1})),\end{aligned} \tag{4.6a}$$

$$\begin{aligned}&\langle x_{n+1}, x_n - x_{n+1}\rangle_1\\ &\geq \langle x_n, x_n - x_{n+1}\rangle_1 - \rho\langle N_1(u_n, v_n) - w_1^*, \eta_1(g_1(x_n), g_1(x_{n+1}))\rangle_1 \\ &\quad + \rho b_1(x_n, g_1(x_{n+1})) - \rho b_1(x_n, g_1(x_n)).\end{aligned} \tag{4.7a}$$

Adding (4.6a) and (4.7a), we have

$$\begin{aligned}&\langle x_n - x_{n+1}, x_{n+1} - x_n\rangle_1\\ &\geq \langle x_{n-1} - x_n, x_{n+1} - x_n\rangle_1\\ &\quad - \rho\langle N_1(u_{n-1} -, v_{n-1}) - N_1(u_n, v_n), \eta_1(g_1(x_{n+1}), g_1(x_n))\rangle_1\\ &\quad + \rho b_1(x_{n-1} - x_n, g_1(x_n)) - \rho b_1(x_n - x_{n-1}, g_1(x_{n+1})),\end{aligned}$$

which implies that

$$
\begin{aligned}
&\|x_n - x_{n+1}\|_1^2 \\
&\le \langle x_{n-1} - x_n, x_n - x_{n+1}\rangle_1 \\
&\quad + \rho\langle N_1(u_{n-1}, v_{n-1}) - N_1(u_n, v_n), \eta_1(g_1(x_{n+1}), g_1(x_n)\rangle_1 \\
&\quad + \rho b_1(x_n - x_{n-1}, g_1(x_n) - g_1(x_{n+1})) \\
&= \langle x_{n-1} - x_n, x_n - x_{n+1} - \rho\eta_1(g_1(x_n), g_1(x_{n+1}))\rangle_1 \\
&\quad + \rho\langle x_{n-1} - x_n - (N_1(u_{n-1}, v_{n-1}) - N_1(u_n, v_n)), \eta_1(g_1(x_n), g_1(x_{n+1}))\rangle_1 \\
&\quad + \rho b_1(x_n - x_{n-1}, g_1(x_n) - g_1(x_{n+1})).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|x_n - x_{n+1}\|_1^2 \\
&\le \|x_{n-1} - x_n\|_1 \|x_n - x_{n+1} - \rho\eta_1(g_1(x_n), g_1(x_{n+1}))\|_1 \\
&\quad + \rho\|x_{n-1} - x_n - (N_1(u_{n-1}, v_{n-1}) - N_1(u_n, v_n))\|_1 \\
&\quad \times \|\eta_1(g_1(x_n), g_1(x_{n+1}))\|_1 \\
&\quad + \rho\gamma_1\|x_n - x_{n-1}\|_1 \|g_1(x_n) - g_1(x_{n+1}))\|_1.
\end{aligned}
\tag{4.8a}
$$

Since η_1 is g_1-strongly monotone with constants $\sigma_1 > 0$, Lipschitz continuous with constants $\delta_1 > 0$ and g_i is Lipschitz continuous with constants $a_1 > 0$, from (4.8a) we have

$$
\begin{aligned}
\|g_1(x_n) - g_1(x_{n+1})\|_1 &\le a_1\|x_n - x_{n+1}\|_1, \\
\|\eta_1(g_1(x_n), g_1(x_{n+1}))\|_1 &\le \delta_1 a_1\|x_n - x_{n+1}\|_1,
\end{aligned}
\tag{4.9a}
$$

and

$$
\begin{aligned}
&\|x_n - x_{n+1} - \rho\eta_1(g_1(x_n), g_1(x_{n+1}))\|_1^2 \\
&\le \|x_n - x_{n+1}\|_1^2 - 2\rho\langle x_n - x_{n+1}, \eta_1(g_1(x_n), g_1(x_{n+1}))\|_1 \\
&\quad + \rho^2\|\eta_1(g_1(x_n), g_1(x_{n+1}))\|_1^2 \\
&\le (1 - 2\rho\sigma_1 + \rho^2\delta_1^2 a_1^2)\|x_n - x_{n+1}\|_1^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|x_n - x_{n+1} - \rho\eta_1(g_1(x_n), g_1(x_{n+1}))\|_1 \\
&\le \sqrt{1 - 2\rho\sigma_1 + \rho^2\delta_1^2 a_1^2}\|x_n - x_{n+1}\|_1.
\end{aligned}
\tag{4.10a}
$$

Since N_1 is strongly Lipschitz continuous in the first argument with constants $\alpha_1 > 0$ and is Lipschitz continuous in the second argument with constants

$\beta_2 > 0$, from (4.8a) we have

$$\begin{aligned}
&\|x_{n-1} - x_n - (N_1(u_{n-1}, v_{n-1}) - N_1(u_n, v_n))\|_1 \\
&\le \|x_{n-1} - x_n - (N_1(u_{n-1}, v_{n-1}) - N_1(u_n, v_{n-1}))\|_1 \\
&\quad + \|N_1(u_n, v_{n-1}) - N_1(u_n, v_n)\|_1 \\
&\le \alpha_1\|x_{n-1} - x_n\|_1 + \beta_2\|v_{n-1} - v_n\|_2 \\
&\le \alpha_1\|x_{n-1} - x_n\|_1 + \beta_2\xi_2\left(1 + \frac{1}{n+1}\right)\|y_{n-1} - y_n\|_2.
\end{aligned} \tag{4.11a}$$

From (4.9a), (4.10a) and (4.11a), we give that

$$\begin{aligned}
&\|x_n - x_{n+1}\|_1 \\
&\le \left(\sqrt{1 - 2\rho\sigma_1 + \rho^2\delta_1^2 a_1^2} + \rho\delta_1 a_1\alpha_1 + \rho\gamma_1 a_1\right)\|x_{n-1} - x_n\|_1 \\
&\quad + \rho\delta_1 a_1\beta_2\xi_2\left(1 + \frac{1}{n+1}\right)\|y_{n-1} - y_n\|_2.
\end{aligned} \tag{4.12a}$$

Taking $s_1 = y_{n+1}$ in (4.3a) and $s_2 = y_n$ in (4.5a), similarly we have

$$\begin{aligned}
&\|y_n - y_{n+1}\|_2 \\
&\le \left(\sqrt{1 - 2\rho\sigma_2 + \rho^2\delta_2^2 a_2^2} + \rho\delta_2 a_2\alpha_2 + \rho\gamma_2 a_2\right)\|y_{n-1} - y_n\|_2 \\
&\quad + \rho\delta_2 a_2\beta_1\xi_1\left(1 + \frac{1}{n+1}\right)\|x_{n-1} - x_n\|_1.
\end{aligned} \tag{4.13a}$$

Combining (4.12a) and (4.13a), we infer

$$\begin{aligned}
&\|x_n - x_{n+1}\|_1 + \|y_n - y_{n+1}\|_2 \\
&\le \Bigg(\sqrt{1 - 2\rho\sigma_1 + \rho^2\delta_1^2 a_1^2} + \rho\delta_1 a_1\alpha_1 \\
&\qquad + \rho\beta_1\xi_1\left[\frac{\gamma_1 a_1}{\beta_1\xi_1} + \delta_2 a_2\left(1 + \frac{1}{n+1}\right)\right]\Bigg)\|x_n - x_{n+1}\|_1 \\
&\quad + \Bigg(\sqrt{1 - 2\rho\sigma_2 + \rho^2\delta_2^2 a_2^2} + \rho\delta_2 a_2\alpha_2 \\
&\qquad + \rho\beta_2\xi_2\left[\frac{\gamma_2 a_2}{\beta_2\xi_2} + \delta_1 a_1\left(1 + \frac{1}{n+1}\right)\right]\Bigg)\|y_n - y_{n+1}\|_2 \\
&= \max\{\theta_{1n}, \theta_{2n}\}(\|x_{n-1} - x_n\|_1 + \|y_{n-1} - y_n\|_2),
\end{aligned} \tag{4.14a}$$

where

$$\theta_{1n} = \sqrt{1 - 2\rho\sigma_1 + \rho^2\delta_1^2 a_1^2} + \rho\delta_1 a_1\alpha_1 + \rho\beta_1\xi_1\left[\frac{\gamma_1 a_1}{\beta_1\xi_1} + \delta_2 a_2\left(1 + \frac{1}{n+1}\right)\right],$$

$$\theta_{2n} = \sqrt{1 - 2\rho\sigma_2 + \rho^2\delta_2^2 a_2^2} + \rho\delta_2 a_2\alpha_2 + \rho\beta_2\xi_2\left[\frac{\gamma_2 a_2}{\beta_2\xi_2} + \delta_1 a_1\left(1 + \frac{1}{n+1}\right)\right].$$

Letting

$$\theta_1 = \sqrt{1 - 2\rho\sigma_1 + \rho^2\delta_1^2 a_1^2} + \rho\delta_1 a_1 \alpha_1 + \rho\beta_1\xi_1 \left[\frac{\gamma_1 a_1}{\beta_1\xi_1} + \delta_2 a_2\right],$$

$$\theta_2 = \sqrt{1 - 2\rho\sigma_2 + \rho^2\delta_2^2 a_2^2} + \rho\delta_2 a_2 \alpha_2 + \rho\beta_2\xi_2 \left[\frac{\gamma_2 a_2}{\beta_2\xi_2} + \delta_1 a_1\right].$$

We can see that $\theta_{1n} \to \theta_1$ and $\theta_{2n} \to \theta_2$ as $n \to \infty$.

Now, defined the norm $\|\cdot\|_*$ on $H_1 \times H_2$ by

$$\|(u, v)\|_* = \|u\|_1 + \|v\|_2 \quad \forall\, (u, v) \in H_1 \times H_2.$$

Observe that $(H_1 \times H_2, \|\cdot\|_*)$ is a Banach space. Hence (4.14a) implies that

$$\|(x_n, y_n) - (x_{n+1}, y_{n+1})\|_* \le \max\{\theta_{1n}, \theta_{2n}\} \|(x_{n-1}, y_{n-1}) - (x_n, y_n)\|_*. \quad (4.15a)$$

For each $i \in I$, according to the condition (4.1a), we have $\theta_i < 1$. Hence, there is a positive number $\theta_0 < 1$ and integer $n_0 \ge 1$ such that $\theta_{in} \le \theta_0 < 1$ for all $n \ge n_0$. Therefore, it follows from (4.15a) that $\{(x_n, y_n)\}$ is a Cauchy sequence in $K_1 \times K_2$. Let $(x_n, y_n) \to (\hat{x}, \hat{y})$ in $K_1 \times K_2$ as $n \to \infty$, since the set-valued mappings A and T are both $\hat{H}$-Lipschitz continuous, from Algorithm 3.2 we get that

$$\|u_n - u_{n+1}\|_1 \le \left(1 + \frac{1}{n+1}\right) \hat{H}(Ax_n, Ax_{n+1}) \le 2\xi_1 \|x_n - x_{n+1}\|_1,$$

$$\|v_n - v_{n+1}\|_2 \le \left(1 + \frac{1}{n+1}\right) \hat{H}(Ty_n, Ty_{n+1}) \le 2\xi_2 \|y_n - y_{n+1}\|_2.$$

Therefore $\{(u_n, v_n)\}$ is also a Cauchy sequence in $H_1 \times H_2$, let $(u_n, v_n) \to (\hat{u}, \hat{v}) \in H_1 \times H_2$ as $n \to \infty$. Noticing $u_n \in Ax_n$, we have

$$\begin{aligned} d(\hat{u}, A\hat{x}) &\le \|\hat{u} - u_n\|_1 + d(u_n, Ax_n) + \hat{H}(Ax_n, A\hat{x}) \\ &\le \|\hat{u} - u_n\|_1 + \xi_1 \|x_n - \hat{x}\|_1 \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

hence $\hat{u} \in A\hat{x}$. Similarly, we can show $\hat{v} \in T\hat{y}$.

Now, we rewrite (3.4) and (3.5) as follows:

$$\begin{aligned} &\langle x_{n+1} - x_n, s_1 - x_{n+1}\rangle_1 + \rho\langle N_1(u_n, v_n) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1 \\ &+ \rho b_1(x_n, g_1(s_1)) - \rho b_1(x_n, g_1(x_{n+1})) \ge 0, \end{aligned} \quad (4.16a)$$

$$\begin{aligned} &\langle y_{n+1} - y_n, s_2 - y_{n+1}\rangle_2 + \rho\langle N_2(u_n, v_n) - w_2^*, \eta_2(g_2(s_2), g_2(y_{n+1}))\rangle_2 \\ &+ \rho b_2(y_n, g_2(s_2)) - \rho b_2(y_n, g_2(y_{n+1})) \ge 0. \end{aligned} \quad (4.17a)$$

Since $(x_n, y_n) \to (\hat{x}, \hat{y}), (u_n, v_n) \to (\hat{u}, \hat{v})$ strongly in $K_1 \times K_2$ and $u_n \in Ax_n$, we have

$$\begin{aligned}
&|\langle N_1(u_n, v_n) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1 \\
&\quad - \langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(\hat{x}))\rangle_1| \\
&\le |\langle N_1(u_n, v_n) - N_1(\hat{u}, \hat{v}), \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1| \\
&\quad + |\langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1})) - \eta_1(g_1(s_1), g_1(\hat{x}))\rangle_1| \\
&\le (\|N_1(u_n, v_n) - N_1(\hat{u}, v_n)\|_1 + \|N_1(\hat{u}, v_n) - N_1(\hat{u}, \hat{v})\|_1) \\
&\quad \times \|\eta_1(g_1(s_1), g_1(x_{n+1}))\|_1 + \|N_1(\hat{u}, \hat{v}) - w_1^*\|_1 \|\eta_1(g_1(x_{n+1}), g_1(\hat{x}))\|_1 \\
&\le (\xi_1 \|u_n - \hat{u}\|_1 + \xi_2 \|v_n - \hat{v}\|_1) \|\eta_1(g_1(s_1), g_1(x_{n+1}))\|_1 \\
&\quad + \delta_1 a_1 \|N_1(\hat{u}, \hat{v}) - w_1^*\|_1 \|x_{n+1} - \hat{x}\|_1 \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Furthermore, from the property of b_1 and Remark 2.1 it follows that

$$\begin{aligned}
&|b_1(x_n, g_1(x_{n+1})) - b_1(\hat{x}, g_1(\hat{x}))| \\
&\le |b_1(x_n, g_1(x_{n+1})) - b_1(x_n, g_1(\hat{x}))| + |b_1(x_n, g_1(\hat{x})) - b_1(\hat{x}, \|)| \\
&\le \gamma_1 a_1 \|x_n\|_1 \|x_{n+1} - \hat{x}\|_1 + \gamma_1 \|x_n - \hat{x}\|_1 \|g_1(\hat{x})\|_1 \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

hence $b_1(x_n, g_1(x_{n+1})) \to b_1(\hat{x}, g_1(\hat{x}))$ and $b_1(x_n, g_1(s_1)) \to b_1(\hat{x}, g_1(s_1))$ as $n \to \infty$.

Let $n \to \infty$ in (4.16a), we obtain

$$\begin{aligned}
&\langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(\hat{x}))\rangle_1 \\
&+ b_1(\hat{x}, g_1(s_1)) - b_1(\hat{x}, g_1(\hat{x})) \ge 0, \quad \forall s_1 \in K_1,
\end{aligned}$$

let $n \to \infty$ in (4.17a), similarly we have

$$\begin{aligned}
&\langle N_2(\hat{u}, \hat{v}) - w_2^*, \eta_2(g_2(s_2), g_2(\hat{y}))\rangle_2 \\
&+ b_2(\hat{y}, g_2(s_2)) - b_2(\hat{y}, g_2(\hat{y})) \ge 0, \quad \forall s_2 \in K_2.
\end{aligned}$$

Therefore $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ is a solutions of the problem (2.1) and (2.2). This completes the proof. □

Theorem 4.2. *For each $i \in I$, let K_i be a nonempty convex subset of H_i and bifunction $b_i(\cdot, \cdot)$ satisfies the conditions* (c1)∼(c4). *Let $N_i : H_1 \times H_2 \to H_i$ be strongly Lipschitz continuous in the first argument and be Lipschitz continuous in the second argument with constants $\alpha_i > 0$ and $\beta_i > 0$, respectively; set-valued mappings $A : K_1 \to CB(H_1)$, $T : K_2 \to CB(H_2)$ be $\hat{H}$-Lipschitz continuous with constants $\xi_1 > 0$ and $\xi_2 > 0$, respectively; linear mapping $g_i : K_i \to K_i$, be Lipschitz continuous with constants $a_i > 0$; and $\eta_i : K_i \times K_i \to H_i$ satisfies Assumption* 2.9 *and η_i be g_i-strongly monotone with constants $\sigma_i > 0$, Lipschitz continuous with constants $\delta_i > 0$. If there exists a constant $\rho > 0$*

such that

$$A_i < 0, \quad B_i^2 - A_iC_i > 0, \quad \left|\rho - \frac{B_i}{A_i}\right| < \frac{\sqrt{B_i^2 - A_iC_i}}{|A_i|}, \tag{4.1b}$$

where

$$A_i = [t_i(\alpha_i + \sqrt{1 - 2b_i + a_i^2}) + c_i\epsilon_i]^2 - t_i^2,$$
$$B_i = a_i^2\sigma_i[t_i(\alpha_i + \sqrt{1 - 2b_i + a_i^2}) + c_i\epsilon_i], \quad C_i = b_i^2 - a_i^2,$$
$$t_i = \delta_i a_i, \quad c_i = b_i\beta_i\xi_i, \quad \epsilon_1 = \frac{\gamma_1 a_1}{c_1} + \frac{t_2}{b_2}, \quad \epsilon_2 = \frac{\gamma_2 a_1}{c_2} + \frac{t_1}{b_1}, \quad i = 1, 2,$$

then there are $(\hat{x}, \hat{y}) \in K_1 \times K_2$, $\hat{u} \in A\hat{x}$, $\hat{v} \in T\hat{y}$ *satisfying the system of generalized nonlinear mixed variational-like inequalities* (2.1) *and* (2.2), *and*

$$(x_n, y_n) \to (\hat{x}, \hat{y}), \quad u_n \to \hat{u}, \quad v_n \to \hat{v}, \quad n \to \infty,$$

where $\{(x_n, y_n)\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$ *are defined by Algorithm* 3.4.

Proof. Using Algorithm 3.4, we obtain that

$$\begin{aligned}&\langle g_1(x_n), s_1 - x_n\rangle_1\\ &\geq \langle g_1(x_{n-1}), s_1 - x_n\rangle_1 - \rho\langle N_1(u_{n-1}, v_{n-1}) - w_1^*, \eta_1(g_1(s_1), g_1(x_n))\rangle_1\\ &\quad + \rho b_1(x_{n-1}, g_1(x_n)) - \rho b_1(x_{n-1}, g_1(s_1)),\end{aligned} \tag{4.2b}$$

$$\begin{aligned}&\langle g_2(y_n), s_2 - y_n\rangle_2\\ &\geq \langle g_2(y_{n-1}), s_1 - y_n\rangle_1 - \rho\langle N_2(u_{n-1}, v_{n-1}) - w_2^*, \eta_2(g_2(s_2), g_2(y_n))\rangle_2\\ &\quad + \rho b_2(y_{n-1}, g_2(y_n)) - \rho b_2(y_{n-1}, g_2(s_2)),\end{aligned} \tag{4.3b}$$

$$\begin{aligned}&\langle g_1(x_{n+1}), s_1 - x_{n+1}\rangle_1\\ &\geq \langle g_1(x_n), s_1 - x_{n+1}\rangle_1 - \rho\langle N_1(u_n, v_n) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1\\ &\quad + \rho b_1(x_n, g_1(x_{n+1})) - \rho b_1(x_n, g_1(s_1)),\end{aligned} \tag{4.4b}$$

$$\begin{aligned}&\langle g_2(y_{n+1}), s_2 - y_{n+1}\rangle_2\\ &\geq \langle g_2(y_n), s_1 - y_{n+1}\rangle_1 - \rho\langle N_2(u_n, v_n) - w_2^*, \eta_2(g_2(s_2), g_2(y_{n+1}))\rangle_2\\ &\quad + \rho b_2(y_n, g_2(y_{n+1})) - \rho b_2(y_n, g_2(s_2))\end{aligned} \tag{4.5b}$$

for all $n \geq 1$. Taking $s_1 = x_{n+1}$ in (4.2b) and $s_1 = x_n$ in (4.4b), we conclude that

$$\begin{aligned}&\langle g_1(x_n), x_{n+1} - x_n\rangle_1\\ &\geq \langle g_1(x_{n-1}), x_{n+1} - x_n\rangle_1 - \rho\langle N_1(u_{n-1}, v_{n-1}) - w_1^*, \eta_1(g_1(x_{n+1}), g_1(x_n))\rangle_1\\ &\quad + \rho b_1(x_{n-1}, g_1(x_n)) - \rho b_1(x_{n-1}, g_1(x_{n+1})),\end{aligned} \tag{4.6b}$$

$$
\begin{aligned}
&\langle g_1(x_{n+1}), x_n - x_{n+1}\rangle_1 \\
&\geq \langle g_1(x_n), x_n - x_{n+1}\rangle_1 - \rho\langle N_1(u_n, v_n) - w_1^*, \eta_1(g_1(x_n), g_1(x_{n+1}))\rangle_1 \\
&\quad + \rho b_1(x_n, g_1(x_{n+1})) - \rho b_1(x_n, g_1(x_n)).
\end{aligned} \tag{4.7b}
$$

Adding (4.6b) and (4.7b), we have

$$
\begin{aligned}
&\langle g_1(x_n) - g_1(x_{n+1}), x_{n+1} - x_n\rangle_1 \\
&\geq \langle g_1(x_{n-1}) - g_1(x_n), x_{n+1} - x_n\rangle_1 \\
&\quad - \rho\langle N_1(u_{n-1}-, v_{n-1}) - N_1(u_n, v_n), \eta_1(g_1(x_{n+1}), g_1(x_n))\rangle_1 \\
&\quad + \rho b_1(x_{n-1} - x_n, g_1(x_n)) - \rho b_1(x_n - x_{n-1}, g_1(x_{n+1})).
\end{aligned}
$$

Since g_1 is strongly monotone with constants $b_1 > 0$, we have

$$
\|g_1(x_n) - g_1(x_{n+1}) - (x_n - x_{n+1})\|_1 \leq \sqrt{1 - 2b_1 + a_1^2}\|x_n - x_{n+1}\|_1, \tag{4.8b}
$$

which implies that

$$
\begin{aligned}
b_1\|x_n - x_{n+1}\|_1^2 &\leq \langle g_1(x_{n-1}) - g_1(x_n), x_n - x_{n+1}\rangle_1 \\
&\quad + \rho\langle N_1(u_{n-1}, v_{n-1}) - N_1(u_n, v_n), \eta_1(g_1(x_{n+1}), g_1(x_n)\rangle_1 \\
&\quad + \rho b_1(x_n - x_{n-1}, g_1(x_n) - g_1(x_{n+1})) \\
&= \langle g_1(x_{n-1}) - g_1(x_n), x_n - x_{n+1} - \rho\eta_1(g_1(x_n), g_1(x_{n+1}))\rangle_1 \\
&\quad + \rho\langle g_1(x_{n-1}) - g_1(x_n) - (N_1(u_{n-1}, v_{n-1}) \\
&\quad - N_1(u_n, v_n)), \eta_1(g_1(x_n), g_1(x_{n+1}))\rangle_1 \\
&\quad + \rho b_1(x_n - x_{n-1}, g_1(x_n) - g_1(x_{n+1})).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&b_1\|x_n - x_{n+1}\|_1^2 \\
&\leq \|g_1(x_{n-1}) - g_1(x_n)\|_1\|x_n - x_{n+1} - \rho\eta_1(g_1(x_n), g_1(x_{n+1}))\|_1 \\
&\quad + \rho\|g_1(x_{n-1}) - g_1(x_n) - (N_1(u_{n-1}, v_{n-1}) - N_1(u_n, v_n))\|_1 \\
&\qquad \times \|\eta_1(g_1(x_n), g_1(x_{n+1}))\|_1 \\
&\quad + \rho\gamma_1\|x_n - x_{n-1}\|_1\|g_1(x_n) - g_1(x_{n+1}))\|_1 \\
&\leq \|g_1(x_{n-1}) - g_1(x_n)\|_1\|x_n - x_{n+1} - \rho\eta_1(g_1(x_n), g_1(x_{n+1}))\|_1 \\
&\quad + \rho\|x_{n-1} - x_n - (N_1(u_{n-1}, v_{n-1}) - N_1(u_n, v_n))\|_1 \\
&\qquad \times \|\eta_1(g_1(x_n), g_1(x_{n+1}))\|_1 + \rho\|g_1(x_{n-1}) - g_1(x_n) - (x_{n-1} - x_n)\|_1 \\
&\quad + \rho\gamma_1\|x_n - x_{n-1}\|_1\|g_1(x_n) - g_1(x_{n+1}))\|_1.
\end{aligned} \tag{4.9b}
$$

From (4.9a), (4.10a), (4.11a) and (4.8b), we give that

$$
\begin{aligned}
&\|x_n - x_{n+1}\|_1 \\
&\le b_1^{-1}\Big(a_1\sqrt{1-2\rho\sigma_1+\rho^2\delta_1^2a_1^2} \\
&\quad + \rho\delta_1 a_1\Big(\alpha_1+\sqrt{1-2b_1+a_1^2}\Big)+\rho\gamma_1 a_1\Big)\|x_{n-1}-x_n\|_1 \\
&\quad + \rho b_1^{-1}\delta_1 a_1\beta_2\xi_2\left(1+\frac{1}{n+1}\right)\|y_{n-1}-y_n\|_2.
\end{aligned}
\tag{4.10b}
$$

Taking $s_1 = y_{n+1}$ in (4.3b) and $s_2 = y_n$ in (4.5b), similarly we have

$$
\begin{aligned}
&\|y_n - y_{n+1}\|_2 \\
&\le b_2^{-1}\Big(a_2\sqrt{1-2\rho\sigma_2+\rho^2\delta_2^2a_2^2} \\
&\quad + \rho\delta_2 a_2\Big(\alpha_2+\sqrt{1-2b_2+a_2^2}\Big)+\rho\gamma_2 a_2\Big)\|y_{n-1}-y_n\|_2 \\
&\quad + \rho b_2^{-1}\delta_2 a_2\beta_1\xi_1\left(1+\frac{1}{n+1}\right)\|x_{n-1}-x_n\|_1.
\end{aligned}
\tag{4.11b}
$$

Combining (4.10b) and (4.11b), we infer

$$
\begin{aligned}
&\max\{\|x_n - x_{n+1}\|_1, \|y_n - y_{n+1}\|_2\} \\
&\le b_1^{-1}\Bigg(a_1\sqrt{1-2\rho\sigma_1+\rho^2\delta_1^2a_1^2}+\rho\delta_1 a_1\Big(\alpha_1+\sqrt{1-2b_1+a_1^2}\Big) \\
&\quad + \rho b_1\beta_1\xi_1\left[\frac{\gamma_1 a_1}{b_1\beta_1\xi_1}+\frac{\delta_2 a_2}{b_2}\left(1+\frac{1}{n+1}\right)\right]\Bigg)\|x_n - x_{n+1}\|_1 \\
&\quad + b_2^{-1}\Bigg(a_2\sqrt{1-2\rho\sigma_2+\rho^2\delta_2^2a_2^2}+\rho\delta_2 a_2\Big(\alpha_2+\sqrt{1-2b_2+a_2^2}\Big) \\
&\quad + \rho b_2\beta_2\xi_2\left[\frac{\gamma_2 a_2}{b_2\beta_2\xi_2}+\frac{\delta_1 a_1}{b_1}\left(1+\frac{1}{n+1}\right)\right]\Bigg)\|y_n - y_{n+1}\|_2 \\
&= \max\{\theta_{1n}, \theta_{2n}\}(\|x_{n-1}-x_n\|_1+\|y_{n-1}-y_n\|_2),
\end{aligned}
\tag{4.12b}
$$

where

$$
\begin{aligned}
\theta_{1n} &= b_1^{-1}\Bigg(a_1\sqrt{1-2\rho\sigma_1+\rho^2\delta_1^2a_1^2}+\rho\delta_1 a_1\Big(\alpha_1+\sqrt{1-2b_1+a_1^2}\Big) \\
&\qquad + \rho b_1\beta_1\xi_1\left[\frac{\gamma_1 a_1}{b_1\beta_1\xi_1}+\frac{\delta_2 a_2}{b_2}\left(1+\frac{1}{n+1}\right)\right]\Bigg), \\
\theta_{2n} &= b_2^{-1}\Bigg(a_2\sqrt{1-2\rho\sigma_2+\rho^2\delta_2^2a_2^2}+\rho\delta_2 a_2\Big(\alpha_2+\sqrt{1-2b_2+a_2^2}\Big) \\
&\qquad + \rho b_2\beta_2\xi_2\left[\frac{\gamma_2 a_2}{b_2\beta_2\xi_2}+\frac{\delta_1 a_1}{b_1}\left(1+\frac{1}{n+1}\right)\right]\Bigg).
\end{aligned}
$$

Letting

$$\theta_1 = b_1^{-1}\Big(a_1\sqrt{1-2\rho\sigma_1+\rho^2\delta_1^2a_1^2}+\rho\delta_1a_1\Big(\alpha_1+\sqrt{1-2b_1+a_1^2}\Big)$$
$$+\rho b_1\beta_1\xi_1\left[\frac{\gamma_1a_1}{b_1\beta_1\xi_1}+\frac{\delta_2a_2}{b_2}\right]\Big),$$
$$\theta_2 = b_2^{-1}\Big(a_2\sqrt{1-2\rho\sigma_2+\rho^2\delta_2^2a_2^2}+\rho\delta_2a_2\Big(\alpha_2+\sqrt{1-2b_2+a_2^2}\Big)$$
$$+\rho b_2\beta_2\xi_2\left[\frac{\gamma_2a_2}{b_2\beta_2\xi_2}+\frac{\delta_1a_1}{b_1}\right]\Big).$$

We can see that $\theta_{1n}\to\theta_1$ and $\theta_{2n}\to\theta_2$ as $n\to\infty$. Now, defined the norm $\|\cdot\|_*$ on $H_1\times H_2$ by

$$\|(u,v)\|_* = \max\{\|u\|_1,\|v\|_2\},\quad \forall\,(u,v)\in H_1\times H_2.$$

It observe that $(H_1\times H_2,\|\cdot\|_*)$ is a Banach space. Hence (4.12b) implies that

$$\|(x_n,y_n)-(x_{n+1},y_{n+1})\|_*\le\max\{\theta_{1n},\theta_{2n}\}\|(x_{n-1},y_{n-1})-(x_n,y_n)\|_*. \quad (4.13b)$$

For each $i\in I$, according to the condition (4.1), we have $\theta_i<1$. Hence, there is a positive number $\theta_0<1$ and integer $n_0\ge1$ such that $\theta_{in}\le\theta_0<1$ for all $n\ge n_0$. Therefore, it follows from (4.15a) that $\{(x_n,y_n)\}$ is a Cauchy sequence in $K_1\times K_2$. Let $(x_n,y_n)\to(\hat{x},\hat{y})$ in $K_1\times K_2$ as $n\to\infty$, since the set-valued mappings A and T are both $\hat{H}$-Lipschitz continuous, from Algorithm 3.4 we get that

$$\|u_n-u_{n+1}\|_1\le\left(1+\frac{1}{n+1}\right)\hat{H}(Ax_n,Ax_{n+1})\le2\xi_1\|x_n-x_{n+1}\|_1,$$

$$\|v_n-v_{n+1}\|_2\le\left(1+\frac{1}{n+1}\right)\hat{H}(Ty_n,Ty_{n+1})\le2\xi_2\|y_n-y_{n+1}\|_2.$$

Therefore $\{(u_n,v_n)\}$ is also a Cauchy sequence in $H_1\times H_2$, let $(u_n,v_n)\to(\hat{u},\hat{v})\in H_1\times H_2$ as $n\to\infty$. Noticing $u_n\in Ax_n$, we have

$$\begin{aligned}d(\hat{u},A\hat{x})&\le\|\hat{u}-u_n\|_1+d(u_n,Ax_n)+\hat{H}(Ax_n,A\hat{x})\\&\le\|\hat{u}-u_n\|_1+\xi_1\|x_n-\hat{x}\|_1\to0\quad\text{as } n\to\infty,\end{aligned}$$

hence $\hat{u}\in A\hat{x}$. Similarly, we can show $\hat{v}\in T\hat{y}$.

Now, we rewrite (3.9) and (3.10) as follows:

$$\begin{aligned}&\langle g_1(x_{n+1})-g_1(x_n),s_1-x_{n+1}\rangle_1\\&+\rho\langle N_1(u_n,v_n)-w_1^*,\eta_1(g_1(s_1),g_1(x_{n+1}))\rangle_1\\&+\rho b_1(x_n,g_1(s_1))-\rho b_1(x_n,g_1(x_{n+1}))\ge0,\end{aligned} \quad (4.14b)$$

$$\begin{aligned}&\langle g_1(y_{n+1}) - g_1(y_n), s_2 - y_{n+1}\rangle_2\\&+ \rho\langle N_2(u_n, v_n) - w_2^*, \eta_2(g_2(s_2), g_2(y_{n+1}))\rangle_2\\&+ \rho b_2(y_n, g_2(s_2)) - \rho b_2(y_n, g_2(y_{n+1})) \geq 0.\end{aligned}\tag{4.15b}$$

Since $x_n \to \hat{x}$ strongly as $n \to \infty$, $\langle g_1(x_{n+1}) - g_1(x_n), s_1 - x_{n+1}\rangle_1 \to 0$ as $n \to \infty$. Note also that

$$\begin{aligned}&\langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(\hat{x}))\rangle_1\\&\geq \limsup_{n\to\infty}\langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1.\end{aligned}$$

Since $N_1(u_n, v_n) \to N_1(\hat{u}, \hat{v})$ strongly in H_1, from Assumption 2.9(3) and boundedness of $\eta_1(g_1(s_1), g_1(x_{n+1}))$, we obtain

$$\begin{aligned}0 &\leq \langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(\hat{x}))\rangle_1\\&\quad - \limsup_{n\to\infty}\langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1\\&= \liminf_{n\to\infty}\{\langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(\hat{x}))\rangle_1\\&\qquad - \langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1\}\\&= \liminf_{n\to\infty}\{\langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(\hat{x}))\rangle_1\\&\qquad - \langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1\\&\qquad + \langle N_1(\hat{u}, \hat{v}) - w_1^* - (N_1(\hat{u}, \hat{v}) - w_1^*), \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1\}\\&= \liminf_{n\to\infty}\{\langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(\hat{x}))\rangle_1\\&\qquad - \langle N_1(u_n, v_n) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1\},\end{aligned}$$

and hence,

$$\begin{aligned}&\langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(\hat{x}))\rangle_1\\&\geq \limsup_{n\to\infty}\langle N_1(u_n, v_n) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1.\end{aligned}$$

Furthermore, from the property of b_1 and Remark 2.1 it follows that

$$\begin{aligned}&|b_1(x_n, g_1(x_{n+1})) - b_1(\hat{x}, g_1(\hat{x}))|\\&\leq |b_1(x_n, g_1(x_{n+1})) - b_1(x_n, g_1(\hat{x}))| + |b_1(x_n, g_1(\hat{x})) - b_1(\hat{x}, \|)|\\&\leq \gamma_1 a_1\|x_n\|_1\|x_{n+1} - \hat{x}\|_1 + \gamma_1\|x_n - \hat{x}\|_1\|g_1(\hat{x})\|_1 \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

hence $b_1(x_n, g_1(x_{n+1})) \to b_1(\hat{x}, g_1(\hat{x}))$ and $b_1(x_n, g_1(s_1)) \to b_1(\hat{x}, g_1(s_1))$ as $n \to \infty$. Therefore, we get

$$\begin{aligned} 0 \leq \limsup_{n\to\infty} \{ & \langle g_1(x_{n+1}) - g_1(x_n), s_1 - x_{n+1}\rangle_1 \\ & + \rho\langle N_1(u_n, v_n) - w_1^*, \eta_1(g_1(s_1), g_1(x_{n+1}))\rangle_1 \\ & + \rho b_1(x_n, g_1(s_1)) - \rho b_1(x_n, g_1(x_{n+1}))\} \\ \leq \rho & \langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(\hat{x}))\rangle_1 \\ & + \rho b_1(\hat{x}, g_1(s_1)) - \rho b_1(\hat{x}, g_1(\hat{x})) \end{aligned} \tag{4.16b}$$

which implies that

$$\langle N_1(\hat{u}, \hat{v}) - w_1^*, \eta_1(g_1(s_1), g_1(\hat{x}))\rangle_1 + b_1(\hat{x}, g_1(s_1)) - b_1(\hat{x}, g_1(\hat{x})) \geq 0, \quad \forall\, s_1 \in K_1.$$

To (4.15b), similarly we have

$$\langle N_2(\hat{u}, \hat{v}) - w_2^*, \eta_2(g_2(s_2), g_2(\hat{y}))\rangle_2 + b_2(\hat{y}, g_2(s_2)) - b_2(\hat{y}, g_2(\hat{y})) \geq 0, \quad \forall\, s_2 \in K_2.$$

Therefore $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ is a solution of the problem (2.1) and (2.2). This completes the proof. □

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